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Robust Stabilization of Uncertain Systems Based on Energy Dissipation Concepts

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Abstract

Robust stability conditions obtained through generalization of the notion of energy dissipation in physical systems are discussed in this report. Linear time-invariant (LTI) systems which dissipate energy corresponding to quadratic power functions are characterized in the time-domain and the frequency-domain, in terms of linear matrix inequalities (LMIs) and algebraic Riccati equations (AREs). A novel characterization of strictly dissipative LTI systems is introduced in this report. Sufficient conditions in terms of dissipativity and strict dissipativity are presented for (1) stability of the feedback interconnection of dissipative LTI systems, (2) stability of dissipative LTI systems with memoryless feedback nonlinearities, and (3) quadratic stability of uncertain linear systems. It is demonstrated that the framework of dissipative LTI systems investigated in this report unifies and extends small gain, passivity and sector conditions for stability. Techniques for selecting power functions for characterization of uncertain plants and robust controller synthesis based on these stability results are introduced. A spring-mass-damper example is used to illustrate the application of these methods for robust controller synthesis.

Keywords: *State space characterization of dissipative LTI systems, stability of interconnected dissipative systems, Linear Matrix Inequalities (LMIs) for robust stability.*

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Chapter 1

Introduction

Intuitively, a system which dissipates energy would eventually lose all of its initial energy, and would approach a configuration corresponding to zero energy if energy is not added to the system. Feedback interconnection of dissipative systems would be stable if the interconnection is such that individual subsystems dissipate their energy. Using mathematical abstractions of the notions of physical power and energy, the concept of energy dissipation has been employed to develop sufficient conditions for stability with dissipative systems. Numerous stability results in the literature such as small gain conditions, passivity conditions, and sector conditions for stability follow naturally as special cases from this framework of dissipative systems.

Energy dissipation in a passive mechanical system is reviewed, first, as an introduction to the characterization of dissipative systems. Consider small oscillations of a passive mechanical system, such as a spring-mass-damper system[1], about its equilibrium configuration. Let $q = \{q_1, \dots, q_n\}^T$ be a vector of n generalized coordinates characterizing the kinematic configuration of this system. These generalized coordinates are selected such that $q = 0$ is an equilibrium configuration of the system. Small oscillations of the system about this equilibrium are being investigated. Let $f = \{f_1, \dots, f_n\}^T$ be the corresponding vector of generalized forces. For small oscillations, kinetic energy of the system is $T = \frac{1}{2}\dot{q}^T M \dot{q}$, where $M = M^T > 0$ is a symmetric, positive definite mass-inertia matrix of the system, and \dot{q} corresponds to generalized velocities. Potential energy of the system is expressed as $V = \frac{1}{2}q^T K q$, where $K = K^T > 0$ is a symmetric, positive definite, stiffness matrix of the sys-

tem. Energy dissipation occurs within the system due to forces proportional to the generalized velocities, which resist the motion of this system [1]. These forces are represented by Rayleigh's dissipation function, $\mathcal{R} = \dot{q}^T D \dot{q}$, where $D = D^T \geq 0$ is a symmetric, positive semidefinite matrix, characterizing damping in the system. Lagrangian equations of motion for natural dynamic systems, including Rayleigh's dissipation function[1, 2], are given as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \left(\frac{\partial L}{\partial q_i} \right) + \left(\frac{\partial \mathcal{R}}{\partial \dot{q}_i} \right) = f_i, \quad i = 1, \dots, n.$$

where L is the Lagrangian, $L = T - V$. This leads to the equations of motion for small oscillations of the mechanical system about its equilibrium configuration as

$$M\ddot{q} + D\dot{q} + Kq = f$$

Natural outputs of this system are generalized velocities, $y = \dot{q}$, such that the dot product (or inner product) of input forces and natural outputs, $y \cdot f$ (equivalently, $y^T f$) gives the power input to the system.

The total energy of this system, E , is the sum of its kinetic energy and its potential energy, $E = T + V = \frac{1}{2} (\dot{q}^T M \dot{q} + q^T K q)$. Note that the total energy is a quadratic, positive definite function of the state of the system. The rate of change of total energy of the system is

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{2} (\ddot{q}^T M \dot{q} + \dot{q}^T M \ddot{q} + \dot{q}^T K q + q^T K \dot{q}) \\ &= \dot{q}^T (M\ddot{q} + Kq) \end{aligned}$$

Substituting for $(M\ddot{q} + Kq)$ from the equations of motion of the system, and using $y = \dot{q}$, results in the power balance equation,

$$\begin{aligned} \frac{dE}{dt} &= \dot{q}^T (f - D\dot{q}) \\ &= y^T f - \dot{q}^T D \dot{q} \end{aligned}$$

This power balance equation mathematically states that the rate of change of total energy of the system is equal to the power input into the system minus the rate of energy dissipation in the system. Since the Rayleigh dissipation function is nonnegative for passive systems ($\mathcal{R} \geq 0$), the power balance equation leads to the following inequality, which is known as the dissipation inequality,

$$\frac{dE}{dt} \leq y^T f \tag{1.1}$$

This inequality states that for a passive mechanical system, rate of change of total energy is less than or equal to power input to the system.

Integrating the power balance equation over an arbitrary interval of time $[t_0, t_1]$ leads to the energy balance equation,

$$\int_{t_0}^{t_1} y^T f dt = \int_{t_0}^{t_1} \mathcal{R} dt + E(t_1) - E(t_0)$$

The energy balance equation expresses the fact that the total energy input to the system over an arbitrary interval of time, given by the time integral of input power over that interval, is equal to the energy dissipated by the system in that time period plus the net change in energy of the system. The energy dissipated by a passive system over any time interval is nonnegative since its integrand, Rayleigh's dissipation function, is nonnegative. Thus, the passive mechanical system satisfies the following integral form of the dissipation inequality,

$$\int_{t_0}^{t_1} y^T f dt \geq E(t_1) - E(t_0) \quad (1.2)$$

over an arbitrary time interval $[t_0, t_1]$, and all admissible inputs f . Admissible inputs are the inputs for which the equations of motion have a well-posed solution. This restriction is placed on the inputs for technical reasons, and is satisfied by most inputs encountered in physical systems. Note that the integral form of the dissipation inequality may be obtained directly by integrating Eq. 1.1) over an arbitrary time interval. It states that the total energy input to the system over an arbitrary time interval is greater than or equal to the net change in energy of the system, the difference being the energy dissipated by the system.

The conditions imposed by energy dissipation in a passive mechanical system are given in differential form by Eq. (1), and in integral form by Eq. (2). In these inequalities, the inner product of the input forces and output velocities, $y^T f$, represents the physical power input to the system, p ; and the total energy function, E , represents physical energy of the system, since it is the sum of the kinetic and potential energies. Consider a generalization of the expression of power to a quadratic function of the input and the output; and that of the energy function to an arbitrary, positive definite, quadratic function of system states. Linear time-invariant (LTI) systems which satisfy the energy dissipation inequality with respect to these generalized power and energy functions are studied in this report. Note that the quadratic power functions

and quadratic energy functions may not have any physical interpretation corresponding to the notions of power and energy in mechanics. However, employing quadratic power functions and energy functions leads to the characterization of a large class of dissipative LTI systems, which includes many types of systems investigated in the literature, such as gain bounded systems, passive systems, and sector bounded systems. Furthermore, parallels from the physical notions of power and energy are exploited to develop stability results for interconnection of dissipative systems. This results in a framework unifying and extending a number of stability results in the literature.

A very general description of dissipative dynamic systems is presented in Refs. [3, 4, 5]. Consider a dynamic system in state space form, $\dot{x} = g(x, f, t)$, $y = h(x, f, t)$, where x denotes the system state, f represents input to the system, y is the system output, and the nonlinear functions, g and h , describe the system dynamics. This system is said to be dissipative, according to Ref. [3], if there exists an absolutely integrable function of the input and the output, the power function, $p(y, f)$, (referred to as the supply rate in Ref. [3]), and a function of the system state, the energy function, $E(x)$, (referred to as the storage function in Ref. [3]), such that

$$\int_{t_0}^{t_1} p(y, f) dt \geq E(x(t_1)) - E(x(t_0)) \quad (1.3)$$

for all admissible inputs, $f(t)$, and arbitrary time intervals $[t_0, t_1]$, $y(t)$ being the response of the dynamic system. The references provide concise results for characterization and stability of general dissipative dynamic systems.

This report restricts attention to linear time-invariant (LTI) systems, which are dissipative with respect to quadratic power functions. This allows the development of specific expressions characterizing dissipative LTI systems, and computational algorithms for determining dissipativity with respect to quadratic power functions[6]. Time-domain and frequency-domain conditions characterizing dissipative LTI systems are developed in terms of linear matrix inequalities (LMIs) and algebraic Riccati equations (AREs). These state space characterizations of dissipative LTI systems are shown to be generalizations of the corresponding characterizations of gain bounded systems, sector bounded systems, as well as positive real systems. Novel concepts of input gain-matrix bounded LTI systems, and output gain-matrix bounded LTI systems are introduced as generalizations of gain bounded LTI systems. These systems are also shown to be dissipative with respect to certain quadratic power functions. Strictly dissipative LTI systems are proposed as a further restricted class of

dissipative LTI systems, and these systems are characterized in time-domain and frequency-domain. Sufficient conditions are presented for (1) stability of the feedback interconnection of dissipative LTI systems, (2) stability of dissipative LTI systems with memoryless feedback nonlinearities, and (3) quadratic stability of uncertain linear systems. Results for the three stability problems mentioned above obtained using small gain, passivity and sector criteria, are shown to be special cases of the results for dissipative LTI systems. New stability results for feedback interconnection of LTI systems, in terms of input/output gain-matrix bounded LTI systems, also follow as special cases of the stability results for dissipative LTI systems. Finally, numerical techniques for tight characterization of plant uncertainty employing convex programming techniques for linear matrix inequalities, are presented; and, an approach for robust dissipative controller synthesis is discussed. A numerical example of a spring-mass-damper system is used to demonstrate the application of the results developed in this report for robust controller synthesis.

Chapter 2

Dissipative LTI Systems

Consider a linear, time-invariant system, Σ , given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bf(t), \\ y(t) &= Cx(t) + Df(t),\end{aligned}\tag{2.1}$$

where $y(t)$ is the $p \times 1$ output vector, $f(t)$ is an $m \times 1$ input vector, $x(t)$ is an $n \times 1$ state vector and the system matrices (A, B, C, D) describe the dynamics of the system. The $p \times m$ transfer function matrix for this system is $G(s) = C(sI - A)^{-1}B + D$. A general quadratic power function of the input and the output is expressed as

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}\tag{2.2}$$

where $Q = Q^T$ is a real symmetric $p \times p$ matrix, $R = R^T$ is a real symmetric $m \times m$ matrix, and N is a real $p \times m$ matrix. The LTI system, Σ , is defined to be dissipative with respect to a quadratic power function, $p(y, f)$, as follows (Ref. [3]).

Definition 2.1 *A stable LTI system, $\Sigma : \dot{x} = Ax + Bf, y = Cx + Df$, where (A, B, C, D) is a minimal system realization, is dissipative with respect to the quadratic power function,*

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

if there exists a positive definite, quadratic energy function, $E(x) = x^T P x$, with $P = P^T > 0$, which satisfies the dissipation inequality

$$\int_0^T p(y, f) dt \geq E(x(T)) - E(x(0)), \quad (2.3)$$

for all $T \in [0, \infty)$ and for all $f \in \mathcal{L}_{2e}^m$.

The extended space of square integrable functions, \mathcal{L}_{2e}^m , is defined in the Appendix. Essentially this space characterizes a set of well-behaved input functions, such that well-posed solutions to the dynamic equations exist, when the input belongs to this space. This space includes almost all signals encountered in practical applications. The condition in Eq. (2.3) is the integral form of the dissipation inequality presented in the introduction. However, note that for the definition above, $t_0 = 0$ and $t_1 = T$ can be used without loss of generality since linear *time-invariant* systems are being considered. This inequality ensures that for a dissipative LTI system, the time integral of power input over any interval, that is, the total energy input to the system, is greater than or equal to the net change in total energy of the system. The difference between the total energy input to the system and the net change in energy of the system is the energy dissipated by the system. This is why systems which satisfy the inequality above are called dissipative systems. Note again that the quadratic power function, $p(y, f)$, and the quadratic energy function, $E(x)$, may not have any physical interpretation, but are mathematical abstractions with properties similar to those of physical power and energy. The dissipativity condition can be expressed in differential form as

$$\frac{d}{dt} E(x) \leq p(y, f) \quad (2.4)$$

by differentiating the expression in Eq. (2.3) with respect to time. In differential form, the dissipation inequality states that the rate of change of the stored energy is less than or equal to the input power, the difference being the rate of energy dissipation.

Observe that if the coefficient matrix for the quadratic power function in Eq. (2.2) is positive semidefinite, that is, if

$$\begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \geq 0$$

it follows from linear regulator theory that the dissipation inequality in Eq. (2.3) is satisfied by all stable systems [7, 8]. Further, if the matrix is negative definite, the

dissipation inequality is not satisfied by any nontrivial system. Thus, dissipative LTI systems are characterized by quadratic power functions whose coefficient matrix is indefinite, that is, it is neither positive semidefinite, nor negative semidefinite. The set of LTI systems which satisfy the dissipation inequality when the coefficient matrix is indefinite is a restricted subset of stable LTI systems which is said to be dissipative with respect to the quadratic power function. The analysis of dissipative LTI systems with quadratic power functions may be considered as an extension of the linear regulator theory[7, 8]. Conditions characterizing dissipativity of LTI systems, with respect to quadratic power functions, appear in terms of linear matrix inequalities and algebraic Riccati equations, parallel to those of linear regulator theory.

Many systems considered in the literature can be treated as special cases of dissipative LTI systems as defined above. In particular, gain bounded systems, positive real systems, and sector bounded systems are dissipative with respect to specified quadratic power functions, as shown below.

Linear time-invariant systems whose \mathcal{H}_∞ norm bounded by unity, (also referred to as bounded real systems in the literature [9, 10]) satisfy

$$\int_0^T y^T(t)y(t)dt \leq \int_0^T f^T(t)f(t)dt$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. Rewriting this condition as

$$\int_0^T \{f^T(t)f(t) - y^T(t)y(t)\}dt \geq 0 \quad (2.5)$$

it is seen that bounded real systems are dissipative LTI systems with a quadratic power function $p(y, f) = f^T(t)f(t) - y^T(t)y(t)$, that is, the general quadratic power function in Eq. (2.2), with $R = I$, $Q = -I$, and $N = 0$.

Similarly, general norm bounded systems with $\|G(s)\|_\infty \leq \gamma$ satisfy

$$\int_0^T y^T(t)y(t)dt \leq \gamma^2 \int_0^T f^T(t)f(t)dt$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. Thus, these systems are dissipative with respect to $p(y, f) = \gamma^2 f^T(t)f(t) - y^T(t)y(t)$, that is, the general quadratic power function in Eq. (2.2) with $R = \gamma^2 I$, $Q = -I$, and $N = 0$.

Passive systems are characterized by the input-output property [9, 11]

$$\int_0^T y^T(t)f(t)dt \geq 0 \quad (2.6)$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. This condition corresponds to dissipativity with respect to a quadratic power function $p(y, f) = y^T(t)f(t) + f^T(t)y(t)$ or the general quadratic power function in Eq. (2.2) with $R = 0, Q = 0$, and $N = I$. In fact, as noted in the first section, dissipative systems are obtained as a generalization of passive systems.

Sector bounded systems also are special cases of dissipative LTI systems. For example, consider an LTI system inside sector $[a, b]$ with $b > 0 > a$. By definition[12, 13], the input and the output of this system satisfy

$$\int_0^T \{y(t) - af(t)\}^T \{y(t) - bf(t)\} dt \leq 0$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. Rewriting this condition as

$$\int_0^T \{-abf^T(t)f(t) + (a+b)y^T(t)f(t) - y^T(t)y(t)\} dt \geq 0 \quad (2.7)$$

shows that the sector bounded LTI system is dissipative with respect to the quadratic power function $p(y, f) = \{-abf^T(t)f(t) + (a+b)y^T(t)f(t) - y^T(t)y(t)\}$ or the general quadratic power function in Eq. (2.2) with $R = -abI, Q = -I$, and $N = \alpha I$, where $\alpha = (a+b)/2$.

These three examples show the generality of the class of LTI systems that are dissipative with respect to quadratic power functions. These classes of systems are obtained as special cases of dissipative LTI systems, simply by substituting specific values for the matrices Q, N and R . Moreover, note the special cases are obtained by substituting scalar matrices in the coefficient matrix for quadratic power functions. On the other hand, the following extensions of the notion of gain bounded LTI systems provide examples of cases where Q and R are full matrices.

An LTI systems, Σ , with transfer function, $G(s) = C(sI - A)^{-1}B + D$ and \mathcal{H}_∞ gain bounded by γ , that is $\|G(s)\|_\infty \leq \gamma$, satisfies $\|y\|_2 \leq \|\gamma f\|_2$, or equivalently, $\int_0^T y^T y dt \leq \int_0^T \gamma^2 f^T f dt$. As an extension to characterization of systems by a positive scalar, its \mathcal{H}_∞ norm, consider MIMO systems being characterized by symmetric, positive definite matrices rather than positive scalar gains. A stable LTI system is defined to be *input gain-matrix bounded* with respect to a symmetric, positive definite matrix, $\gamma_i = \gamma_i^T > 0$, if $\|y\|_2 \leq \|\gamma_i f\|_2$, or equivalently, if $\int_0^T y^T y dt \leq \int_0^T f^T \gamma_i f dt$, for all $f \in \mathcal{L}_{2e}^m$. Input gain-matrix bounded systems are easily seen to be dissipative with respect to the quadratic power functions with $Q = -I, R = \gamma_i$, and $N = 0$.

To see input gain-matrix bounded systems as an extension of gain bounded systems, note that an LTI system whose gain is bounded by γ is input gain-matrix bounded with respect to the scalar matrix, $\gamma_i = \gamma I$. General positive definite values for γ_i can be employed for a tighter characterization of LTI systems, as discussed later in this report.

Similarly, noting that a gain bounded LTI system also satisfies $\|\frac{1}{\gamma}y\|_2 \leq \|f\|_2$, a stable LTI system is defined as *output gain-matrix bounded* with respect to a symmetric positive definite matrix, $\gamma_o = \gamma_o^T > 0$, if $\|\gamma_o^{-1}y\|_2 \leq \|f\|_2$, or equivalently, if $\int_0^T y^T \gamma_o^{-2} y dt \leq \int_0^T f^T f dt$ for all $f \in \mathcal{L}_{2e}^m$. Output gain-matrix bounded LTI systems are seen to be dissipative with respect to quadratic power functions with $R = I$, $Q = -\gamma_o^{-2}$, and $N = 0$. Note that an LTI system whose gain is bounded by γ is output gain-matrix bounded with respect to the scalar gain matrix $\gamma_o = \gamma I$.

Combining these two ideas, an LTI system is input-output gain-matrices bounded with respect to symmetric positive definite matrices $\gamma_i = \gamma_i^T > 0$ and $\gamma_o = \gamma_o^T > 0$, if $\|\gamma_o^{-1}y\|_2 \leq \|\gamma_i f\|_2$, or equivalently, $\int_0^T y^T \gamma_o^{-2} y dt \leq \int_0^T f^T \gamma_i^2 f dt$ for all $f \in \mathcal{L}_{2e}^m$. This system is seen to be dissipative with respect to a quadratic power function with $R = \gamma_i^2$, $Q = -\gamma_o^{-2}$, and $N = 0$. It should be noted that the gain matrices γ_i and γ_o are not independent of each other in the sense that if a system is input-output gain-matrices bounded with respect to γ_i and γ_o , then it is also bounded with respect to $\alpha \gamma_i$ and $\frac{1}{\alpha} \gamma_o$, where α is any positive scalar. Again, to see that these systems form a generalization of gain bounded systems, note that a system whose gain is bounded by γ is input-output gain-matrices bounded with respect to scalar matrices $\gamma_i = \gamma_i I$ and $\gamma_o = \gamma_o I$ with $\gamma = \gamma_i \gamma_o$.

Frequency-domain characterization of gain-matrix bounded LTI systems in a later section will further clarify the notion that constraints imposed by the definitions above are generalizations of gain bounded systems. Moreover, stability results for gain-matrix bounded systems will be obtained from the general stability result for dissipative LTI systems in later sections.

Chapter 3

Time-Domain Characterization

Time-domain characterizations of dissipative LTI systems are developed in this section. The first characterization is presented in terms of a constrained solution of a system of three matrix equations. This characterization is referred to as the dissipativity lemma, since it is a generalization of the Kalman-Yakubovich Lemma (or positive realness lemma) for positive real systems, and the bounded realness lemma for gain bounded systems[9, 10]. The conditions of the dissipativity lemma can be equivalently expressed in terms of a linear matrix inequality (LMI). The LMI characterization of dissipative LTI systems is very important in application of these results for tight characterization of uncertain plants in terms of dissipativity, as described in later sections. LMI characterizations of gain bounded, positive real, and sector bounded systems[6] follow directly from the LMI characterization of dissipative LTI systems by substituting their respective power functions.

State space characterization of LTI systems which are dissipative with respect to quadratic power functions is presented in the following Theorem.

Theorem 3.1 *Consider a stable LTI system, $\Sigma : \dot{x} = Ax + Bf, y = Cx + Df$, where (A, B, C, D) is a minimal realization of the system. The following statements are equivalent.*

1. The LTI system, Σ , is dissipative with respect to a quadratic power function,

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

2. (Dissipativity Lemma) There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy the following system of three matrix equations,

$$\begin{aligned} PA + A^T P &= C^T Q C - L^T L \\ PB &= C^T (QD + N) - L^T W \\ R + N^T D + D^T N + D^T Q D &= W^T W \end{aligned} \quad (3.1)$$

3. (LMI Characterization) There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality

$$\begin{bmatrix} C^T Q C - PA - A^T P & C^T (QD + N) - PB \\ (QD + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} \geq 0 \quad (3.2)$$

Proof: (1) \Leftarrow (2) : Assuming that there exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the conditions of the dissipativity lemma, Eq. (3.1), it is shown that $E(x) = x^T P x$ is a quadratic energy function which satisfies the dissipativity inequality in differential form, Eq. (2.4). Consider

$$\begin{aligned} \frac{d}{dt} E(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T (A^T P + P A) x + f^T B^T P x + x^T P B f \end{aligned}$$

Using the first two relations in Eq. (3.1) gives

$$\begin{aligned} \frac{d}{dt} E(x) &= -x^T L^T L x + x^T C^T Q C x + f^T (QD + N)^T C x + \\ &\quad x^T C^T (QD + N) f - f^T W^T L x - x^T L^T W f \end{aligned}$$

Adding and subtracting $f^T W^T W f$ for “completing the square” leads to

$$\begin{aligned} \frac{d}{dt} E(x) &= -x^T L^T L x - x^T L^T W f - f^T W^T L x - f^T W^T W f \\ &\quad + x^T C^T Q C x + x^T C^T Q D f + f^T D^T Q C x \\ &\quad + x^T C^T N f + f^T N^T C x + f^T W^T W f \end{aligned}$$

Using the last relation in Eq. (3.1) to substitute for $f^T W^T W f$ gives

$$\begin{aligned} \frac{d}{dt}E(x) = & -x^T L^T (Lx + Wf) - f^T W^T (Lx + Wf) \\ & + x^T C^T Q (Cx + Df) + f^T D^T Q (Cx + Df) \\ & (x^T C^T + f^T D^T) N f + f^T N^T (Cx + Df) + f^T R f \end{aligned}$$

Thus, using $y = Cx + Df$, it follows that

$$\frac{d}{dt}E(x) = -(Lx + Wf)^T (Lx + Wf) + p(y, f) \quad (3.3)$$

Since $(Lx + Wf)^T (Lx + Wf) \geq 0$, for any input, $f \in \mathcal{L}_{2e}^m$, the differential form of the dissipativity inequality, $\frac{d}{dt}E(x) \leq p(y, f)$, follows. Integrating this inequality over an arbitrary time interval, $[0, T]$, leads to the integral form of the dissipation inequality in Eq. (2.3).

(2) \Leftarrow (3) : Let M be a Cholesky factor of the matrix in the linear matrix inequality condition, Eq. (3.2), that is,

$$\begin{bmatrix} C^T Q C - P A - A^T P & C^T (Q D + N) - P B \\ (Q D + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} = M^T M \geq 0$$

Partitioning M as $M = \begin{bmatrix} L & W \end{bmatrix}$ conformally to the partitions on left-hand side, leads to

$$\begin{bmatrix} C^T Q C - P A - A^T P & C^T (Q D + N) - P B \\ (Q D + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} = \begin{bmatrix} L^T L & L^T W \\ W^T L & W^T W \end{bmatrix} \quad (3.4)$$

Conditions of the dissipativity lemma in Eq. (3.1) follow by equating the submatrices in Eq. (3.4).

(2) \Rightarrow (3) : This follows by reversing the steps above. Forming the matrix in the LMI condition, Eq. (3.2), using the conditions of the dissipativity lemma, Eq. (3.1), leads to

$$\begin{bmatrix} C^T Q C - P A - A^T P & C^T (Q D + N) - P B \\ (Q D + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} = \begin{bmatrix} L^T L & L^T W \\ W^T L & W^T W \end{bmatrix} \quad (3.5)$$

The right-hand side of Eq. (3.5) may be written as $M^T M$, with $M = \begin{bmatrix} L & W \end{bmatrix}$. Thus, the right-hand side of Eq. (3.5) is nonnegative definite, which is the LMI condition of (3).

(1) \Rightarrow (3) : The dissipation inequality in differential form, Eq. (2.4), ensures that there exists a quadratic energy function $E(x) = x^T P x$, with $P = P^T > 0$, which satisfies $p(y, f) - \frac{d}{dt}E(x) \geq 0$. Algebraic manipulations of this conditions, shown below, lead to the LMI condition in (3).

First, expand the quadratic power function in terms of the state and the input as follows.

$$\begin{aligned}
p(y, f) &= y^T Q y + y^T N f + f^T N^T y + f^T R f \\
&= (Cx + Df)^T Q (Cx + Df) + (Cx + Df)^T N f + f^T N^T (Cx + Df) + f^T R f \\
&= x^T C^T Q C x + x^T C^T (QD + N) f + f^T (QD + N)^T C x \\
&\quad + f^T (D^T Q D + D^T N + N^T D + R) f \\
&= \begin{bmatrix} x^T & f^T \end{bmatrix} \begin{bmatrix} C^T Q C & C^T (QD + N) \\ (QD + N)^T C & R + N^T D + D^T N + D^T Q D \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} \quad (3.6)
\end{aligned}$$

Further, express the derivative of the energy function, $\frac{d}{dt}E(x)$, as a quadratic in terms of the state and the output as follows.

$$\begin{aligned}
\frac{d}{dt}E(x) &= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T (A^T P + P A) x + f^T B^T P x + x^T P B f \\
&= \begin{bmatrix} x^T & f^T \end{bmatrix} \begin{bmatrix} P A + A^T P & P B \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} \quad (3.7)
\end{aligned}$$

Substituting from Eq. (3.7) and Eq. (3.6) into the differential form of the dissipation inequality gives

$$\begin{bmatrix} x^T & f^T \end{bmatrix} \begin{bmatrix} C^T Q C - P A - A^T P & C^T (QD + N) - P B \\ (QD + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} \geq 0$$

Since the dissipation inequality must be valid for all $f \in \mathcal{L}_{2e}^m$, and controllability of the system realization implies that the inequality must be satisfied for all x , it follows that

$$\begin{bmatrix} C^T Q C - P A - A^T P & C^T (QD + N) - P B \\ (QD + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} \geq 0$$

This is the LMI characterization of dissipative LTI systems in (3). \square

Theorem 3.1 presents constraints on the system matrices of a minimal realization of an LTI system for the system to be dissipative with respect to a given quadratic power

function. The generality of the results in Theorem 3.1 for dissipative LTI systems is emphasized by noting that the corresponding results for gain bounded systems, positive real systems, and sector bounded systems, follow simply by substituting their respective power functions in the general results.

The bounded realness lemma for characterizing unity gain bounded systems, or bounded real systems[9, 10] and the LMI form of this condition[14, 15] follow by substituting the power function for bounded real systems, that is, setting $R = I$, $Q = -I$, and $N = 0$, in the results of Theorem 3.1.

Corollary 3.1 *Consider a stable LTI system, Σ , with a minimal realization, (A, B, C, D) . The following statements are equivalent.*

1. *The LTI system, Σ , is bounded real.*
2. *(Bounded Realness Lemma) There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying*

$$\begin{aligned} PA + A^T P &= -C^T C - L^T L \\ PB &= -C^T D - L^T W \\ I - D^T D &= W^T W \end{aligned} \tag{3.8}$$

3. *There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality*

$$\begin{bmatrix} PA + A^T P + C^T C & C^T D + PB \\ D^T C + B^T P & D^T D - I \end{bmatrix} \leq 0 \tag{3.9}$$

The Kalman-Yakubovich lemma, also known as the positive realness lemma, for characterizing positive real LTI systems[10, 16], and the LMI characterization of positive real systems[17] follow directly from the results of Theorem 3.1 by substituting the power function for positive systems, that is, by setting $N = I$, $Q = 0$ and $R = 0$.

Corollary 3.2 *Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) . The following statements are equivalent.*

1. *The LTI system, Σ , is positive real.*

2. (Positive Realness Lemma) There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ D + D^T &= W^T W \end{aligned} \tag{3.10}$$

3. There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality

$$\begin{bmatrix} PA + A^T P & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \leq 0 \tag{3.11}$$

State space characterization of LTI systems inside sector $[a, b]$ is presented in terms of sector boundedness lemma and the corresponding linear matrix inequality in Ref [6, 18]. These characterizations follow directly from the results of Theorem 3.1 by setting $R = -abI$, $N = \alpha I$ and $Q = -I$ where $\alpha = (a + b)/2$.

Corollary 3.3 Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) . The following statements are equivalent.

1. The LTI system, Σ , is inside sector $[a, b]$.
2. (Sector Boundedness Lemma.) There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying

$$\begin{aligned} PA + A^T P &= -C^T C - L^T L \\ PB &= C^T(\alpha I - D) - L^T W \\ -abI + \alpha(D + D^T) - D^T D &= W^T W \end{aligned} \tag{3.12}$$

where $\alpha = (a + b)/2$.

3. There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality

$$\begin{bmatrix} PA + A^T P + C^T C & PB - C^T(\alpha I - D) \\ B^T P - (\alpha I - D)^T C & abI - \alpha(D + D^T) + D^T D \end{bmatrix} \leq 0 \tag{3.13}$$

State space characterization of other sector bounded LTI systems in terms of LMIs can be obtained in a similar fashion by substituting respective power functions.

Next three corollaries present the LMI characterization of gain-matrix bounded LTI systems. Again, these results follow simply by substituting their respective power functions into the result of Theorem 3.1. For input gain-matrix bounded systems, the following result is established by setting $Q = -I$, $N = 0$, and $R = \begin{smallmatrix} 2 \\ i \end{smallmatrix}$.

Corollary 3.4 *Consider a stable LTI system, Σ , with a minimal realization, (A, B, C, D) . The following statements are equivalent.*

1. *The LTI system, Σ , is an input gain-matrix bounded system, with respect to a symmetric, positive definite matrix, $\begin{smallmatrix} 2 \\ i \end{smallmatrix}$.*
2. *There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying*

$$\begin{aligned} PA + A^T P &= -C^T C - L^T L \\ PB &= -C^T D - L^T W \\ \begin{smallmatrix} 2 \\ i \end{smallmatrix} - D^T D &= W^T W \end{aligned} \tag{3.14}$$

3. *There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality*

$$\begin{bmatrix} PA + A^T P + C^T C & C^T D + PB \\ D^T C + B^T P & D^T D - \begin{smallmatrix} 2 \\ i \end{smallmatrix} \end{bmatrix} \leq 0 \tag{3.15}$$

A similar result follows for output gain-matrix bounded LTI systems with respect to a symmetric positive definite matrix, $\begin{smallmatrix} 2 \\ o \end{smallmatrix}$, by setting $Q = -\begin{smallmatrix} 2 \\ o \end{smallmatrix}$, $N = 0$, and $R = I$.

Corollary 3.5 *Consider a stable LTI system, Σ , with a minimal realization, (A, B, C, D) . The following statements are equivalent.*

1. *The LTI system, Σ , is output gain-matrix bounded with respect to a symmetric, positive definite matrix, $\begin{smallmatrix} 2 \\ o \end{smallmatrix}$.*

2. (Bounded Realness Lemma) There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying

$$\begin{aligned} PA + A^T P &= -C^T, \quad {}_{\circ}^{-2}C - L^T L \\ PB &= -C^T, \quad {}_{\circ}^{-2}D - L^T W \\ I - D^T, \quad {}_{\circ}^{-2}D &= W^T W \end{aligned} \quad (3.16)$$

3. There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality

$$\begin{bmatrix} PA + A^T P + C^T, \quad {}_{\circ}^{-2}C & C^T, \quad {}_{\circ}^{-2}D + PB \\ D^T, \quad {}_{\circ}^{-2}C + B^T P & D^T, \quad {}_{\circ}^{-2}D - I \end{bmatrix} \leq 0 \quad (3.17)$$

Finally, the result for input-output gain-matrices bounded LTI systems with respect to a symmetric positive definite matrices, γ_i , and γ_o , follows by setting $Q = -\gamma_o^{-2}$, $N = 0$, and $R = \gamma_i^{-2}$.

Corollary 3.6 Consider a stable LTI system, Σ , with a minimal realization, (A, B, C, D) . The following statements are equivalent.

1. The LTI system, Σ , is input-output gain-matrices bounded with respect to a symmetric, positive definite matrices, γ_i , and γ_o .
2. There exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying

$$\begin{aligned} PA + A^T P &= -C^T, \quad {}_{\circ}^{-2}C - L^T L \\ PB &= -C^T, \quad {}_{\circ}^{-2}D - L^T W \\ \gamma_i^{-2} - D^T, \quad {}_{\circ}^{-2}D &= W^T W \end{aligned} \quad (3.18)$$

3. There exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the following linear matrix inequality

$$\begin{bmatrix} PA + A^T P + C^T, \quad {}_{\circ}^{-2}C & C^T, \quad {}_{\circ}^{-2}D + PB \\ D^T, \quad {}_{\circ}^{-2}C + B^T P & D^T, \quad {}_{\circ}^{-2}D - \gamma_i^{-2} \end{bmatrix} \leq 0 \quad (3.19)$$

In concluding this section, it is noted that the LMI characterization of dissipative LTI systems in Theorem 3.1 is a very useful result in applications. Dissipativity of an LTI system with respect to a given power function may be posed as a feasibility problem with the LMI in Eq. (3.2). Moreover, tight characterization of plants can be developed as of optimization of linear objective functions with LMI constraints using the result of Theorem 3.1. Efficient convex programming algorithms are available [19] for the solution of such LMI problems. These techniques will be discussed in greater detail in a later section. The point emphasized here is that the LMI characterization of dissipative LTI systems in Theorem 3.1 is essential for enabling such analysis.

Chapter 4

Algebraic Riccati Equation Characterization

The linear matrix inequality characterizing dissipative LTI systems is equivalent to quadratic matrix inequalities (QMIs), or algebraic Riccati inequalities (ARIs) under an additional constraint. Extremal solutions of the ARIs can be determined from solutions of the corresponding algebraic Riccati equations (AREs)[20, 21, 22]. This leads to the characterization of dissipative LTI systems in terms of algebraic Riccati equations. Again, the ARE characterizations of gain bounded systems, positive real systems, and sector bounded systems follows directly by substituting their respective quadratic power functions[6].

The first result presents characterization of certain dissipative LTI systems with algebraic Riccati inequalities.

Theorem 4.1 *Consider a stable LTI system, $\Sigma : \dot{x} = Ax + Bf$, $y = Cx + Df$, where (A, B, C, D) is a minimal realization of the system, and a quadratic power function,*

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

Assume that the matrix $\tilde{R} = (R + N^T D + D^T N + D^T Q D)$ is positive definite. Then, Σ is dissipative with respect to this quadratic power function if and only if there exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the algebraic

Riccati inequality $\tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^T \geq 0$, where $\tilde{Q} = C^TQC - PA - A^TP$ and $\tilde{N} = C^T(QD + N) - PB$.

Proof: The result follows by using the Schur complements identity to show that the algebraic Riccati inequality above is equivalent to the linear matrix inequality in Theorem 3.1, under positive definiteness of \tilde{R} . Note that rearranging terms, $\tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^T \geq 0$ can be written as

$$C^T \left(Q - (QD + N)\tilde{R}^{-1}(QD + N)^T \right) C - P\tilde{A} - \tilde{A}^TP - PB\tilde{R}^{-1}B^TP \geq 0 \quad (4.1)$$

where $\tilde{A} = A - B\tilde{R}^{-1}(QD + N)^TC$. In this form it is clear that this condition is a quadratic matrix inequality in P .

Since $\tilde{R} > 0$, and $\tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^T \geq 0$,

$$\begin{bmatrix} \tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^T & 0 \\ 0 & \tilde{R} \end{bmatrix} \geq 0$$

Post-multiplying by the nonsingular matrix $T = \begin{bmatrix} I & 0 \\ \tilde{R}^{-1}\tilde{N}^T & I \end{bmatrix}$, and pre-multiplying by T^T gives

$$\begin{bmatrix} I & \tilde{N}\tilde{R}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{Q} - \tilde{N}\tilde{R}^{-1}\tilde{N}^T & 0 \\ 0 & \tilde{R} \end{bmatrix} \begin{bmatrix} I & 0 \\ \tilde{R}^{-1}\tilde{N}^T & I \end{bmatrix} = \begin{bmatrix} \tilde{Q} & \tilde{N} \\ \tilde{N}^T & \tilde{R} \end{bmatrix} \geq 0$$

Thus, with $\tilde{R} > 0$, a symmetric, positive definite matrix, $P = P^T > 0$, satisfies the algebraic Riccati inequality if and only if it satisfies the LMI. Thus, the result follows from Theorem 3.1. \square

Feasibility of the algebraic Riccati inequality can be determined from the solutions of the corresponding algebraic Riccati equation, using comparative theorems for these solutions, and results for extremal solutions of algebraic Riccati equations[21, 22]. Furthermore, conditions for the existence of a symmetric, positive definite solution to algebraic Riccati equations have been studied extensively, in terms of conditions on eigenvalues of the corresponding Hamiltonian matrix[23]. These conditions can be used to establish dissipativity of LTI systems, as summarized in the following Theorem.

Theorem 4.2 Consider a stable LTI system, Σ , with a minimal system realization (A, B, C, D) , and a quadratic power function,

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

Assume that the matrix $\tilde{R} = (R + N^T D + D^T N + D^T Q D)$ is positive definite. Then the system is dissipative with respect to this quadratic power function, if and only if there exists a symmetric, positive definite solution, $P = P^T > 0$ to the algebraic Riccati equation

$$\tilde{A}^T P + P \tilde{A} + P B \tilde{R}^{-1} B^T P - C^T [Q - (QD + N) \tilde{R}^{-1} (QD + N)^T] C = 0, \quad (4.2)$$

where $\tilde{A} = A - B \tilde{R}^{-1} (QD + N)^T C$. Equivalently, the Hamiltonian matrix

$$H = \begin{bmatrix} \tilde{A} & B \tilde{R}^{-1} B^T \\ C^T [Q - (QD + N) \tilde{R}^{-1} (QD + N)^T] C & -\tilde{A}^T \end{bmatrix}$$

does not have eigenvalues on the imaginary axis.

Algebraic Riccati equation and Hamiltonian matrix characterization of the special cases follow by substituting their power functions in the results of Theorem 4.1 and Theorem 4.2. These results are well known for gain bounded systems and positive real systems[22, 24]. The following corollaries demonstrate that these results follow directly from the general result for dissipative systems by substituting the power functions for bounded real and positive real systems.

Corollary 4.1 Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) . Assume the matrix $\tilde{R} = (I - D^T D)$ is symmetric and positive definite. Then, the system is bounded real if and only if there exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the algebraic Riccati inequality

$$A^T P + P A + (C^T D + P B) \tilde{R}^{-1} (C^T D + P B)^T + C^T C \leq 0 \quad (4.3)$$

Equivalently, Σ is bounded real if and only if there exists a symmetric positive definite solution to the algebraic Riccati equation

$$\tilde{A}^T P + P \tilde{A} + P B \tilde{R}^{-1} B^T P + C^T (I + D \tilde{R}^{-1} D^T) C = 0 \quad (4.4)$$

where $\tilde{A} = A + B\tilde{R}^{-1}D^T C$, or if the Hamiltonian matrix

$$H = \begin{bmatrix} A + B\tilde{R}^{-1}D^T C & B\tilde{R}^{-1}B^T \\ -C^T(I + D\tilde{R}^{-1}D^T)C & -(A + B\tilde{R}^{-1}D^T C)^T \end{bmatrix}$$

does not have eigenvalues on the imaginary axis.

Corollary 4.2 Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) . Assume the matrix $\tilde{R} = (D + D^T)$ is symmetric and positive definite. Then, the system is positive real if and only if there exists a symmetric, positive definite matrix, $P = P^T > 0$, which satisfies the algebraic Riccati inequality

$$A^T P + P A + (P B - C^T)\tilde{R}^{-1}(P B - C^T)^T \leq 0 \quad (4.5)$$

Equivalently, Σ is positive real if and only if there exists a symmetric, positive definite solution to the algebraic Riccati equation

$$\tilde{A}^T P + P \tilde{A} + P B \tilde{R}^{-1} B^T P + C^T \tilde{R}^{-1} C = 0, \quad (4.6)$$

where $\tilde{A} = A - B\tilde{R}^{-1}C$, or if the Hamiltonian matrix

$$H = \begin{bmatrix} A - B\tilde{R}^{-1}C & B\tilde{R}^{-1}B^T \\ -C^T\tilde{R}^{-1}C & -(A - B\tilde{R}^{-1}C)^T \end{bmatrix}$$

does not have eigenvalues on the imaginary axis.

Algebraic Riccati equation characterization of sector bounded and gain-matrix bounded LTI systems follows from Theorem 4.2 by substituting their respective quadratic power functions.

Algebraic Riccati equation and Hamiltonian matrix characterization of dissipative LTI systems allows the use of well established numerical techniques to determine whether a given LTI system is dissipative with respect to specified quadratic power functions. This provides an alternative to the LMI technique presented in the previous section for establishing dissipativity of an LTI system with respect to a quadratic power function. Both approaches have their merits. Though efficient convex programming techniques are available for determining the feasibility of LMIs, the algebraic Riccati equation approach is computationally faster for high order systems. However,

the algebraic Riccati equation approach can be used only when \tilde{R} is well-conditioned and can be inverted reliably. In general, it has been observed that numerical accuracy of the LMI approach is superior when the problem data is not well conditioned. Further, LMIs are very useful in determining power functions such that all plants from a given uncertainty set of plants are dissipative with respect to that power function. Thus, it is concluded that both LMI and ARE characterizations of dissipative LTI systems are useful in characterizing dissipative LTI systems.

Chapter 5

Frequency-Domain Properties

Frequency-domain properties of dissipative LTI systems are examined in this section. A frequency-domain inequality that is satisfied by all dissipative LTI systems is presented first. Then, it is shown that this inequality establishes dissipativity with respect to quadratic power functions that have negative semidefinite coefficient matrix Q . In other words, the frequency-domain conditions are necessary and sufficient for power functions with $Q = Q^T \leq 0$, and are necessary for all dissipative LTI systems. Note the class of dissipative systems with $Q = Q^T \leq 0$ includes all the special cases being considered and numerous generalizations. Thus, frequency-domain characterization of gain bounded[9, 10], positive real, and sector bounded systems[18, 12] follows from the results of this section by substituting their respective power functions. Further, frequency-domain conditions for gain-matrix bounded systems are also presented.

The first Theorem presents the frequency-domain inequality satisfied by all dissipative LTI systems.

Theorem 5.1 *If a stable LTI system, Σ , with transfer function matrix, $G(s)$, is dissipative with respect to a quadratic power function*

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

then

$$\phi(j\omega) = \begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \geq 0 \quad (5.1)$$

for all ω .

Proof: Let (A, B, C, D) be a minimal realization of Σ . Since Σ is dissipative with respect to $p(y, f)$, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy the following conditions of the dissipativity lemma,

$$\begin{aligned} PA + A^T P &= C^T Q C - L^T L \\ PB &= C^T (QD + N) - L^T W \\ R + N^T D + D^T N + D^T Q D &= W^T W \end{aligned}$$

Using these relations, the result follows from algebraic manipulations outlined below. Since $G(j\omega) = C(j\omega I - A)^{-1}B + D$, and $G^*(j\omega) = B^T(-j\omega I - A^T)^{-1}C^T + D^T$,

$$\begin{aligned} \phi(j\omega) &= \begin{bmatrix} B^T(-j\omega I - A^T)^{-1}C^T + D^T \end{bmatrix} Q \begin{bmatrix} C(j\omega I - A)^{-1}B + D \end{bmatrix} \\ &\quad + \begin{bmatrix} B^T(-j\omega I - A^T)^{-1}C^T + D^T \end{bmatrix} N + N^T \begin{bmatrix} C(j\omega I - A)^{-1}B + D \end{bmatrix} + R \end{aligned}$$

Expanding and collecting the terms

$$\begin{aligned} \phi(j\omega) &= B^T(-j\omega I - A^T)^{-1}C^T Q C(j\omega I - A)^{-1}B \\ &\quad + B^T(-j\omega I - A^T)^{-1}C^T(QD + N) + (QD + N)^T C(j\omega I - A)^{-1}B \\ &\quad + D^T Q D + D^T N + N^T D + R \end{aligned}$$

Substituting for the relations of the dissipativity lemma, Eq. (3.1),

$$\begin{aligned} \phi(j\omega) &= B^T(-j\omega I - A^T)^{-1} \left(PA + A^T P + L^T L \right) (j\omega I - A)^{-1}B \\ &\quad + B^T(-j\omega I - A^T)^{-1} (PB + L^T W) \\ &\quad + (B^T P + W^T L)(j\omega I - A)^{-1}B + W^T W \end{aligned}$$

Rearranging terms,

$$\begin{aligned} \phi(j\omega) &= B^T(-j\omega I - A^T)^{-1} L^T L(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1} L^T W \\ &\quad + W^T L(j\omega I - A)^{-1}B + W^T W \\ &\quad + B^T(-j\omega I - A^T)^{-1} PB + B^T P(j\omega I - A)^{-1}B \\ &\quad + B^T(-j\omega I - A^T)^{-1} (PA + A^T P)(j\omega I - A)^{-1}B \end{aligned}$$

Note that

$$\begin{aligned}
& B^T(-j\omega I - A^T)^{-1}(-PA - A^T P)(j\omega I - A)^{-1}B \\
&= B^T(-j\omega I - A^T)^{-1} \left[(-j\omega I - A^T)P + P(j\omega I - A) \right] (j\omega I - A)^{-1}B \\
&= B^T P(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}PB
\end{aligned}$$

Thus,

$$\begin{aligned}
\phi(j\omega) &= \left(L(j\omega I - A)^{-1}B + W \right)^* \left(L(j\omega I - A)^{-1}B + W \right) \\
&= \mathcal{D}^*(j\omega)\mathcal{D}(j\omega)
\end{aligned}$$

where $\mathcal{D}(j\omega) = L(j\omega I - A)^{-1}B + W$. Since $\mathcal{D}^*(j\omega)\mathcal{D}(j\omega) \geq 0$ for all ω , the result follows. \square

The next Theorem establishes sufficiency of the frequency-domain inequality for dissipativity of LTI systems with respect to quadratic power functions with negative semidefinite coefficient matrix, Q .

Theorem 5.2 *Consider a stable LTI system, Σ , with transfer function matrix, $G(s)$, and a quadratic power function,*

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

with $Q = Q^T \leq 0$. If the LTI system, Σ , satisfies the frequency-domain condition

$$\phi(j\omega) = \begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \geq 0$$

for all ω , then Σ is dissipative with respect to the quadratic power function, $p(y, f)$.

Proof: The details of this proof are involved, but the key idea is that the matrix relations of the dissipativity lemma follow by comparing minimal realizations on either side of a spectral factorization identity for ϕ [25, 26]. The proof is a generalization of the frequency-domain proofs for bounded realness lemma and positive realness lemma.

First, recall some notation and results reviewed in the Appendix. Paraconjugate of a rational transfer function matrix, $M(s)$, is $M^\sim(s) \equiv M^T(-s)$. If (A, B, C, D) is

a realization of $M(s)$, then $(-A^T, -C^T, B^T, D^T)$ is a realization of $M^\sim(s)$. A matrix $\phi(s)$ is said to be parahermitian if $\phi^\sim(s) = \phi(s)$. Finally, the spectral factorization theorem states that a parahermitian matrix, $\phi(s)$, which is positive semidefinite on the imaginary axis can be factored as $\phi(s) = M^\sim(s)M(s)$, where the spectral factor, $M(s)$, is proper, stable, and its transmission zeros are in the closed left-half plane.

Note that

$$\phi(s) \equiv \begin{bmatrix} G^\sim(s) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(s) \\ I \end{bmatrix}$$

is parahermitian. Further, $\phi(s)$ is assumed to be positive, semidefinite on the imaginary axis. Thus, the spectral factorization theorem ensures that there exists a stable, proper, spectral factor, $M(s)$, of $\phi(s)$, that is,

$$\phi(s) = \begin{bmatrix} G^\sim(s) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(s) \\ I \end{bmatrix} = M^\sim(s)M(s) \quad (5.2)$$

The matrix relations of the dissipativity lemma are obtained by comparing minimal realizations on either side of the spectral factorization identity, Eq. (5.2). Recall from Theorem 3.1 that satisfying the conditions of the dissipativity lemma implies that the LTI system is dissipative with respect to the corresponding quadratic power function.

Let (A_1, B_1, C_1, D_1) be a minimal realization of $M(s)$. Then, $M^\sim(s)$ has a realization $(-A_1^T, -C_1^T, B_1^T, D_1^T)$. Therefore, the product of these transfer function matrices, $M^\sim(s)M(s)$ has a realization

$$\left(\begin{bmatrix} A_1 & 0 \\ -C_1^T C_1 & -A_1^T \end{bmatrix}, \begin{bmatrix} B_1 \\ -C_1^T D_1 \end{bmatrix}, \begin{bmatrix} D_1^T C_1 & B_1^T \end{bmatrix}, D_1^T D_1 \right)$$

Since the state space realization of $M(s)$ is minimal, (A_1, C_1) is observable, and there exists a symmetric positive definite matrix, $X_1 = X_1^T > 0$, such that

$$A_1^T X_1 + X_1 A_1 + C_1^T C_1 = 0 \quad (5.3)$$

Applying the state transformation $\begin{bmatrix} I & 0 \\ X_1 & I \end{bmatrix}$ gives another realization for $M^\sim(s)M(s)$ as

$$\left(\begin{bmatrix} A_1 & 0 \\ 0 & -A_1^T \end{bmatrix}, \begin{bmatrix} B_1 \\ -X_1 B_1 - C_1^T D_1 \end{bmatrix}, \begin{bmatrix} B_1^T X_1 + D_1^T C_1 & B_1^T \end{bmatrix}, D_1^T D_1 \right) \quad (5.4)$$

Note that this realization is minimal since all modes of $M(s)$ are stable, and all modes of $M^\sim(s)$ are unstable.

Next, a minimal realization of the left hand side of the spectral factorization identity in Eq. (5.2) is obtained. Let (A, B, C, D) be a minimal realization of $G(s)$. Then, a realization of $QG(s)$ is (A, B, QC, QD) , that of $G^\sim(s)$ is $(-A^T, -C^T, B^T, D^T)$, and a realization of $G^\sim(s)QG(s)$ is

$$\left(\begin{bmatrix} A & 0 \\ -C^T QC & -A^T \end{bmatrix}, \begin{bmatrix} B \\ -C^T QD \end{bmatrix}, \begin{bmatrix} D^T QC & B^T \end{bmatrix}, D^T QD \right)$$

Since A is stable and $Q = Q^T \leq 0$, there exists a symmetric, positive semidefinite matrix, $X = X^T \geq 0$, such that

$$A^T X + XA = C^T QC \quad (5.5)$$

Applying the state transformation $\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ gives another realization for $G^\sim(s)QG(s)$ as

$$\left(\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ XB - C^T QD \end{bmatrix}, \begin{bmatrix} -B^T X + D^T QC & B^T \end{bmatrix}, D^T QD \right)$$

Further, a minimal realization of $G^\sim(s)N + N^T G(s)$ is

$$\left(\begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}, \begin{bmatrix} B \\ -C^T N \end{bmatrix}, \begin{bmatrix} N^T C & B^T \end{bmatrix}, N^T D + D^T N \right)$$

Thus, a realization of $\phi(s) = G^\sim(s)QG(s) + G^\sim(s)N + N^T G(s) + R$ is (A_r, B_r, C_r, D_r) , where

$$\begin{aligned} A_r &= \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} \\ B_r &= \begin{bmatrix} B \\ XB - C^T(QD + N) \end{bmatrix} \\ C_r &= \begin{bmatrix} -B^T X + (QD + N)^T C & B^T \end{bmatrix} \\ D_r &= R + N^T D + D^T N + D^T QD \end{aligned} \quad (5.6)$$

Note again that this is a minimal realization, since modes of A are stable, and modes of $-A^T$ are unstable.

Now compare the minimal realizations on either side of the spectral factorization identity, Eq. (5.2), given by Eq. (5.4) and Eq. (5.6). It follows that there exists a nonsingular state transformation matrix, T , such that

$$\begin{aligned} A_1 &= T^{-1}AT \\ B_1 &= T^{-1}B \\ B_1^T X_1 + D_1^T C_1 &= \left(-B^T X + (QD + N)^T C\right) T \\ D_1^T D_1 &= R + N^T D + D^T N + D^T QD \end{aligned} \quad (5.7)$$

Detailed arguments for guaranteeing the existence of such a transformation matrix, T , follow from the same arguments as those presented in Ref. [27].

Finally, manipulations with the matrix relations in Eq. (5.7) leads to the relations of the dissipativity lemma. Setting $W = D_1$ gives

$$R + N^T D + D^T N + D^T QD = W^T W$$

Substituting $A_1 = T^{-1}AT$ in Eq. (5.3), premultiplying by T^{-T} and postmultiplying by T^{-1} , results in

$$A^T(T^{-T}X_1T^{-1}) + (T^{-T}X_1T^{-1})A = -T^{-T}C_1^T C_1 T^{-1} \quad (5.8)$$

Adding Eq. (5.5) and Eq. (5.8), and setting $P = X + T^{-T}X_1T^{-1} > 0$ and $L = C_1T^{-1}$, results in

$$A^T P + P A = C^T Q C - L^T L$$

Further, using the second and third relations in Eq. (5.7) leads to

$$B^T(T^{-T}X_1T^{-1}) + D_1^T C_1 T^{-1} = (QD + N)^T C - B^T X$$

Rearranging the terms, and substituting for $P = P^T > 0$, L , and W gives the remaining relation of the dissipativity lemma,

$$PB = C^T(QD + N) - L^T W$$

Thus, it follows from the hypothesis of the theorem that there exists a positive definite matrix, $P = P^T > 0$, and matrices L and W such that

$$\begin{aligned} PA + A^T P &= C^T Q C - L^T L \\ PB &= C^T(QD + N) - L^T W \\ R + N^T D + D^T N + D^T QD &= W^T W \end{aligned}$$

It follows from Theorem 3.1 that the system, Σ , is dissipative with respect to the quadratic power function, $p(y, f)$. \square

Theorem 5.1 and Theorem 5.2 establish that the frequency-domain inequality in Eq. (5.1) is necessary and sufficient to characterize dissipative LTI systems with respect to a quadratic power function for which $Q = Q^T \leq 0$. This condition is satisfied for the special classes of dissipative LTI systems being considered. Therefore, necessary and sufficient frequency-domain conditions characterizing gain bounded, positive real, and sector bounded systems follow by substituting their power functions into the results of Theorems 5.1 and 5.2, and are summarized in the corollaries below. Note that often these equivalent frequency-domain characterizations are used to define these special classes of dissipative systems.

Corollary 5.1 *Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) , and transfer function matrix, $G(s)$. The LTI system is bounded real if and only if its frequency response, $G(j\omega)$, satisfies*

$$\phi(j\omega) = I - G^*(j\omega)G(j\omega) \geq 0 \quad (5.9)$$

for all ω . Equivalently, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy

$$\begin{aligned} PA + A^T P &= -C^T C - L^T L \\ PB &= -C^T D - L^T W \\ I - D^T D &= W^T W \end{aligned}$$

if and only if the frequency-domain inequality in Eq. (5.9) is satisfied for all ω .

Corollary 5.2 *Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) , and transfer function matrix, $G(s)$. The LTI system is positive real if and only if its frequency response, $G(j\omega)$, satisfies*

$$\phi(j\omega) = G^*(j\omega) + G(j\omega) \geq 0 \quad (5.10)$$

for all ω . Equivalently, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ D^T + D &= W^T W \end{aligned}$$

if and only if the frequency-domain inequality in Eq. (5.10) is satisfied for all ω .

Corollary 5.3 Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) , and transfer function matrix, $G(s)$. The LTI system is inside sector $[a, b]$, for $a < 0 < b$, if and only if its frequency response, $G(j\omega)$, satisfies

$$\phi(j\omega) = -abI + \alpha(G^*(j\omega) + G(j\omega)) - G^*(j\omega)G(j\omega) \geq 0 \quad (5.11)$$

for all ω , where $\alpha = (a+b)/2$. Equivalently, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy

$$\begin{aligned} PA + A^T P &= -C^T C - L^T L \\ PB &= C^T(\alpha I - D) - L^T W \\ -abI + \alpha(D^T + D) - D^T D &= W^T W \end{aligned}$$

if and only if the frequency-domain inequality in Eq. (5.11) is satisfied for all ω .

These results also provide the necessary and sufficient frequency-domain conditions for gain-matrix bounded LTI systems. As with gain bounded systems, these equivalent frequency-domain conditions may also be considered as definitions of gain-matrix bounded systems. Also, these conditions further clarify the interpretation of gain-matrix bounded systems as extensions of gain bounded systems. The frequency-domain conditions for these systems follow directly by substituting their power functions into the general results for dissipative LTI systems.

First, consider input gain-matrix bounded LTI system, with respect to the input gain-matrix, γ_i . The corollary below follows by setting $Q = -I$, $N = 0$, and $R = \gamma_i^2$.

Corollary 5.4 Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) , and transfer function matrix, $G(s)$. The LTI system is input gain-matrix bounded with respect to a symmetric positive definite matrix, γ_i , if and only if its frequency response, $G(j\omega)$, satisfies

$$\phi(j\omega) = \gamma_i^2 - G^*(j\omega)G(j\omega) \geq 0 \quad (5.12)$$

for all ω . Equivalently, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy

$$\begin{aligned} PA + A^T P &= -C^T C - L^T L \\ PB &= -C^T D - L^T W \\ \gamma_i^2 - D^T D &= W^T W \end{aligned}$$

if and only if the frequency-domain inequality in Eq. (5.12) is satisfied for all ω .

Similarly, frequency-domain conditions for output gain-matrix bounded systems follow by setting $Q = -\frac{2}{\sigma}, N = 0$, and $R = I$.

Corollary 5.5 *Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) , and transfer function matrix, $G(s)$. The LTI system is output gain-matrix bounded with respect to a symmetric positive definite matrix, $\frac{2}{\sigma}$, if and only if its frequency response, $G(j\omega)$, satisfies*

$$\phi(j\omega) = I - G^*(j\omega), \frac{2}{\sigma} G(j\omega) \geq 0 \quad (5.13)$$

for all ω . Equivalently, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy

$$\begin{aligned} PA + A^T P &= -C^T, \frac{2}{\sigma} C - L^T L \\ PB &= -C^T, \frac{2}{\sigma} D - L^T W \\ I - D^T, \frac{2}{\sigma} D &= W^T W \end{aligned}$$

if and only if the frequency-domain inequality in Eq. (5.13) is satisfied for all ω .

Results for input-output gain-matrices bounded LTI systems with respect to symmetric positive definite matrices, $\frac{2}{i}$ and $\frac{2}{\sigma}$, follow by setting $Q = -\frac{2}{\sigma}, N = 0$, and $R = \frac{2}{i}$.

Corollary 5.6 *Consider a stable LTI system, Σ , with a minimal realization (A, B, C, D) , and transfer function matrix, $G(s)$. The LTI system is input-output gain-matrices bounded with respect to symmetric positive definite matrices, $\frac{2}{i}$ and $\frac{2}{\sigma}$, if and only if its frequency response, $G(j\omega)$, satisfies*

$$\phi(j\omega) = \frac{2}{i} - G^*(j\omega), \frac{2}{\sigma} G(j\omega) \geq 0 \quad (5.14)$$

for all ω . Equivalently, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy

$$\begin{aligned} PA + A^T P &= -C^T, \frac{2}{\sigma} C - L^T L \\ PB &= -C^T, \frac{2}{\sigma} D - L^T W \\ \frac{2}{i} - D^T, \frac{2}{\sigma} D &= W^T W \end{aligned}$$

if and only if the frequency-domain inequality in Eq. (5.14) is satisfied for all ω .

Again, to see that gain-matrix bounded systems are extensions of gain bounded systems, consider an LTI system whose \mathcal{H}_∞ norm is bounded by γ . It is straightforward to check from the frequency-domain conditions that this system is input gain-matrix bounded with respect to the matrix $\gamma_i I$; it is output gain-matrix bounded with respect to the matrix $\gamma_o I$; and it is input-output gain-matrix bounded with respect to the matrices $\gamma_i I$ and $\gamma_o I$, where γ_i and γ_o are scalars such that $\gamma = \gamma_i \gamma_o$. With the scalar structure of the symmetric, positive definite gain-matrices above, gain-matrix bounded systems are reduced to the usual gain bounded systems. However, when these matrices are not restricted to a scalar structure, but are allowed to be general symmetric, positive definite matrices, gain-matrix bounded systems characterize a much larger class LTI systems. Stability results for gain-matrix bounded systems are also derived from those of general dissipative systems and are presented in later sections.

The frequency-domain inequality characterization provides insight into the frequency-domain behavior of the dissipative LTI systems, and this interpretation is very useful in selecting quadratic power functions for uncertain systems. Frequency-domain conditions for positive real systems have been used in the literature for graphically determining positive realness of a system. However, with the efficient numerical methods for LMIs and AREs available now, frequency-domain conditions are not used for determining dissipativity of LTI systems. However, these conditions are very useful in exhibiting the frequency-domain characteristics of dissipative LTI systems, and are often used to define the special cases of dissipative systems.

Chapter 6

Strictly Dissipative Systems

A key notion for the stability of interconnected dissipative systems is that of strictly dissipative systems. As will be seen in the next section, dissipativity of LTI systems establishes Lyapunov stability of interconnected dissipative systems only. A more restrictive notion than dissipativity is needed to establish asymptotic stability of the interconnected dissipative systems. However, the literature on dissipative systems[3, 4, 5] lacks a consistent approach to the notion of strictly dissipative systems. In this section, a novel characterization of strictly LTI dissipative systems is presented, which has been motivated by the requirements of stability of interconnected dissipative systems. State space characterization of strictly dissipative LTI systems, in terms of further constraints beyond the dissipativity lemma, are also presented. Frequency-domain implication of the additional constraints is that the frequency-domain inequality (FDI) must be satisfied in a strict sense. It is shown that this definition of strict dissipativity is consistent with strict bounded realness and strict positive realness.

Consider the energy balance equation for LTI systems which are dissipative with respect to quadratic power functions to explore further restrictions required for strictly dissipative systems,

$$\int_0^T p(y(t), f(t))dt = \int_0^T \mathcal{R}dt + E(T) - E(0) \quad (6.1)$$

where $\int_0^T \mathcal{R}dt$ represents the energy dissipated by the system. The definition for dissipativity requires that the dissipated energy, $\int_0^T \mathcal{R}dt$, be greater than or equal to

zero, so that the dissipation inequality in Eq. (2.3), namely,

$$\int_0^T p(y(t), f(t))dt \geq E(T) - E(0)$$

is satisfied. However, this definition allows dissipative systems to have motion along which no energy is dissipated. That is, it is possible to have $\int_0^T \mathcal{R}dt = 0$ along certain nontrivial state trajectories for dissipative LTI systems. For these state trajectories, the energy balance equation reads $\int_0^T p(y(t), f(t))dt = E(T) - E(0)$, or energy input is equal to the net change in energy of the system, and no energy being dissipated.

Strictly dissipative LTI systems are defined below as systems which dissipate energy along almost all state trajectories of the system. A finite number of trajectories along which no energy dissipation occurs are exponentially stable trajectories, with exponentially decreasing input. An exponentially decreasing input, $f(t)$, is an input of the form $f(t) = \mu(t)e^{st}$, where $\mu(t)$ is a polynomial vector in t of the same dimension as f , and $\text{Re}\{s\} < 0$. It is shown later in this section that these exceptional trajectories correspond to stable transmission zeros of a real rational transfer function matrix. In other words, energy must be dissipated by a strictly dissipative LTI system for almost all motion of the system, and energy dissipation may be equal to zero only for a finite number of exponentially stable trajectories of the system. The proposed definition of strictly dissipative LTI systems based on these consideration is as follows.

Definition 6.1 *A stable LTI system, $\Sigma : \dot{x} = Ax + Bf, y = Cx + Df$, where (A, B, C, D) is a minimal realization, is strictly dissipative with respect to a quadratic power function,*

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

if there exists a positive definite, quadratic energy function, $E(x) = x^T Px$, with $P = P^T > 0$, which satisfies the strict dissipation inequality

$$\int_0^T p(y, f)dt > E(x(T)) - E(x(0)) \quad (6.2)$$

for all $T \in (0, \infty)$ and for all nonzero $f \in \mathcal{L}_{2e}^m$, except for a finite number of exponentially decreasing inputs.

State space characterization of strictly dissipative LTI systems is developed next. Note that a strictly dissipative LTI system is obviously dissipative with respect to

its quadratic power function; however, it satisfies additional constraints beyond those required by dissipativity. Thus, a strictly dissipative system must satisfy the dissipativity lemma, and some additional constraints, presented in the next Theorem.

Theorem 6.1 *A stable LTI system, $\Sigma : \dot{x} = Ax + Bf, y = Cx + Df$, where (A, B, C, D) is a minimal realization of the system, is strictly dissipative with respect to a quadratic power function,*

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix} \quad (6.3)$$

if and only if there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W satisfying the dissipativity lemma,

$$\begin{aligned} PA + A^T P &= C^T Q C - L^T L \\ PB &= C^T (QD + N) - L^T W \\ R + N^T D + D^T N + D^T Q D &= W^T W \end{aligned}$$

the matrices (A, L) are observable, and all transmission zeros of the transfer function, $\mathcal{D}(s) = L(sI - A)^{-1}B + W$, are in the open left-half plane.

Proof: Since a strictly dissipative system also satisfies the conditions for dissipativity, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W , which satisfy the dissipativity lemma. Thus, from the proof of Theorem 3.1 it follows by integrating Eq. (3.3), over the interval $[0, T]$, that

$$\int_0^T p(y(t), f(t)) dt = \int_0^T d^T(t) d(t) dt + E(T) - E(0)$$

where $d = Lx + Wf$. Thus, the rate of energy dissipation function, $\mathcal{R}(t)$ in Eq. (6.1), is given as $\mathcal{R}(t) = d^T(t) d(t)$.

First assume that the LTI system is strictly dissipative. If (A, L) is not observable, it is possible to have nonzero state trajectories in the unobservable subspace of (A, L) along which $d = Lx + Wf$ is identically zero. Therefore, exponentially increasing input can be generated, which excites unobservable states of the system only, such that $\mathcal{R}(t) = d^T(t) d(t) \equiv 0$. The system does not dissipate energy along these unstable trajectories with exponentially increasing input, which implies that the system is not strictly dissipative, a contradiction. Thus, (A, L) must be observable. Since (A, B)

is controllable by the minimality assumption for Σ , it follows that (A, B, L, W) is a minimal realization for $\mathcal{D}(s) = L(sI - A)^{-1}B + W$. Furthermore, $d = Lx + Wf$ is identically zero when the system input is along the direction corresponding to a transmission zero of $\mathcal{D}(s)$. If $\mathcal{D}(s)$ has a transmission zero in the closed right-half plane, then there exists an input f corresponding to this transmission zero, that is not exponentially decreasing, for which $d = Lx + Wf = 0$ and $\mathcal{R}(t) = d^T(t)d(t) \equiv 0$. This again contradicts the hypothesis for a strictly dissipative LTI system, therefore, all transmission zeros of $\mathcal{D}(s)$ must be stable. Thus, for strictly dissipative LTI systems, there must exist a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W , which satisfy the dissipativity lemma, the matrices (A, L) are observable, and all the transmission zeros of $\mathcal{D}(s)$ must be stable.

Conversely, assume that there exists a symmetric, positive definite matrix $P = P^T > 0$, and matrices L and W , which satisfy the dissipativity lemma, (A, L) is observable, and all the transmission zeros of $\mathcal{D}(s)$ are in the open left-half plane. Then, $d = Lx + Wf \equiv 0$ only for input $f \in \mathcal{L}_{2e}^m$, corresponding to stable transmission zeros of $\mathcal{D}(s)$, that is, only for a finite number of exponentially decreasing input. For all other $f \in \mathcal{L}_{2e}^m$, the energy dissipated, $\int_0^T d^T(t)d(t)dt > 0$, that is, the system is strictly dissipative with respect to $p(y, f)$. \square

Note from the proof of Theorem 6.1 that a finite number of system inputs for which a strictly dissipative LTI system does not dissipate energy correspond to stable transmission zeros of $\mathcal{D}(s) = L(sI - A)^{-1}B + W$. For all other inputs $f \in \mathcal{L}_{2e}^m$, a strictly dissipative LTI system must dissipate energy, that is, $\int_0^T \mathcal{R}(t)dt = \int_0^T d^T(t)d(t)dt > 0$.

Further, it should be noted that the differential form of the dissipation inequality in Eq. (2.4), namely, $\frac{d}{dt}E(x) \leq p(y, f)$, is not strengthened to $\frac{d}{dt}E(x) < p(y, f)$, for strictly dissipative LTI systems. It is possible for $d = Lx + Wf$ to be equal to zero at certain time instants, even for strictly dissipative LTI systems. However, $d(t)$ must not be identically equal to zero over a finite time interval. In terms of the matrices L and W , satisfying the dissipativity lemma, this implies that L and W do not have to be full column rank for strict dissipativity. The necessary and sufficient conditions on L and W for strict dissipativity are that (A, L) is observable, and transmission zeros of $\mathcal{D}(s) = L(sI - A)^{-1}B + W$ are stable.

For the frequency-domain conditions, recall from Theorem 5.1 that a dissipative

LTI system satisfies

$$\phi(j\omega) = \begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = \mathcal{D}^*(j\omega)\mathcal{D}(j\omega)$$

for all ω , where $\mathcal{D}(s) = L(sI - A)^{-1}B + W$. Since $\mathcal{D}(s)$ is minimal and does not have transmission zeros on the imaginary axis (its transmission zeros are stable), it follows that $\mathcal{D}^*(j\omega)\mathcal{D}(j\omega) > 0$ for all ω . Thus, if an LTI system with transfer function $G(s)$ is strictly dissipative with respect to a quadratic power function, $p(y, f)$, then its frequency response, $G(j\omega)$, satisfies

$$\phi(j\omega) = \begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} > 0 \quad (6.4)$$

for all ω .

The converse is true for the case where $Q = Q^T \leq 0$ is symmetric, negative semidefinite. In this case, if the frequency response, $G(j\omega)$, of an LTI system satisfies

$$\phi(j\omega) = \begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} > 0$$

for all ω , then from Theorem 5.2 it follows that there exist matrices $P = P^T > 0, L$ and W , which satisfy the dissipativity lemma, so that

$$\phi(j\omega) = \begin{bmatrix} G^*(j\omega) & I \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = \mathcal{D}^*(j\omega)\mathcal{D}(j\omega) > 0$$

for all ω , where $\mathcal{D}(s) = L(sI - A)^{-1}B + W$. Spectral factorization, $\phi(j\omega) = \mathcal{D}^*(j\omega)\mathcal{D}(j\omega)$ can always be performed such that $\mathcal{D}(s)$ is minimum phase and stable with (A, B, L, W) being a minimal realization. This implies that (A, L) is observable and $\mathcal{D}(s)$ does not have transmission zeros in the closed right-half plane. Thus, the strict frequency-domain inequality in Eq. (6.4) ensures strict dissipativity of the LTI system when $Q = Q^T \leq 0$.

For $Q = Q^T \leq 0$, the strict frequency-domain inequality of Eq. (6.4) is a necessary and sufficient condition for strict dissipativity of the LTI system. This observation shows that the definition of strict dissipativity is consistent with strict bounded realness and strict positive realness. Substituting the power function for bounded realness shows that a system with transfer function, $G(s)$, is strictly bounded real

if its frequency response satisfies $\phi(j\omega) = I - G^*(j\omega)G(j\omega) > 0$, for all ω , that is, if $\|G(s)\|_\infty < 1$. This is the definition for strictly bounded real LTI systems in Ref. [9, 10]. Similarly, using the power function for positive real systems in the definition of strictly dissipative systems shows that an LTI system is strictly positive real if and only if $\phi(j\omega) = G^*(j\omega) + G(j\omega) > 0$, for all ω . Again this definition is consistent with the definition of strictly positive real systems in Ref. [28]. Finally, substituting the power function for systems inside sector $[a, b]$ leads to the frequency-domain characterization of LTI systems to be strictly inside sector $[a, b]$ as

$$\phi(j\omega) = -abI + \alpha (G^*(j\omega) + G(j\omega)) - G^*(j\omega)G(j\omega) > 0$$

for all ω , where $\alpha = (a + b)/2$ as in Refs. [12, 18].

Frequency-domain criteria for LTI systems which are strictly gain-matrix bounded follow by substituting their respective power functions. A stable LTI system is strictly input gain-matrix bounded with respect to a symmetric positive definite matrix, $\mathcal{G}_i = \mathcal{G}_i^T > 0$, if and only if $\phi(j\omega) = \mathcal{G}_i^2 - G^*(j\omega)G(j\omega) > 0$ for all ω . A stable LTI system is strictly output gain matrix bounded with respect to a symmetric positive definite matrix, $\mathcal{G}_o = \mathcal{G}_o^T > 0$, if and only if its frequency response satisfies $\phi(j\omega) = I - G^*(j\omega)\mathcal{G}_o^{-2}G(j\omega) > 0$ for all ω . Finally, a stable LTI system is strictly input-output gain-matrix bounded, with respect to matrices, $\mathcal{G}_i = \mathcal{G}_i^T > 0$, and $\mathcal{G}_o = \mathcal{G}_o^T > 0$, if and only if its frequency response satisfies $\phi(j\omega) = \mathcal{G}_i^2 - G^*(j\omega)\mathcal{G}_o^{-2}G(j\omega) > 0$ for all ω . These frequency-domain conditions are necessary and sufficient, and can be used to define these strictly gain matrix bounded systems.

Chapter 7

Stability of Feedback Interconnection

A central result of this report, namely, sufficient conditions for closed-loop stability of the standard feedback interconnection of dissipative LTI systems, is presented in this section. This stability result is shown to be a very general result, in that small gain conditions, passivity conditions, and a number of sector conditions for stability[12, 29] follow as special cases of this result. Further, new stability conditions are obtained for gain-matrix bounded LTI systems by substituting their quadratic power functions, demonstrating the generality of the result for dissipative systems.

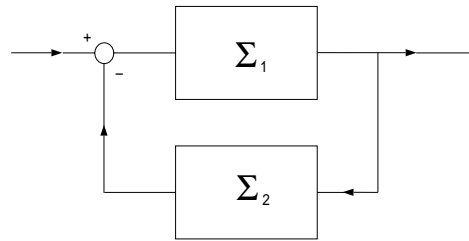


Figure 7.1: Negative Feedback Interconnection of Dissipative LTI Systems.

Consider two stable linear, time-invariant systems, Σ_1 and Σ_2 , in the standard negative feedback interconnection, as shown in Fig. 7.1. Assume that (A_i, B_i, C_i, D_i) are minimal realizations of these systems, such that their dynamics are given by the

state space equations,

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i f_i \\ y_i &= C_i x_i + D_i f_i, \quad i = 1, 2 \end{aligned} \quad (7.1)$$

The standard negative feedback interconnection imposes the conditions $f_2 = y_1$, and $f_1 = -y_2$. Further, assume that these two systems are dissipative with respect to quadratic power functions respectively,

$$p_i(y_i, f_i) = \begin{bmatrix} y_i^T & f_i^T \end{bmatrix} \begin{bmatrix} Q_i & N_i \\ N_i^T & R_i \end{bmatrix} \begin{bmatrix} y_i \\ f_i \end{bmatrix}, \quad i = 1, 2 \quad (7.2)$$

The following Theorem gives sufficient conditions on the power functions under which the feedback interconnection is stable.

Theorem 7.1 *If there exist positive scalars, $\alpha_1 > 0$ and $\alpha_2 > 0$, such that*

$$\alpha_1 p_1(y_1, f_1) + \alpha_2 p_2(-f_1, y_1) \leq 0 \quad (7.3)$$

for all y_1 and f_1 , then the standard feedback interconnection of dissipative LTI systems, Σ_1 and Σ_2 , is Lyapunov stable. Furthermore, if either of these systems is strictly dissipative, then the closed-loop system is asymptotically stable.

Proof: A weighted sum of the energy functions of these two dissipative systems is used as a Lyapunov function to establish the stability results.

Since the LTI systems, Σ_i , for $i = 1, 2$ are dissipative with respect to quadratic power functions, $p_i(y_i, f_i)$, by Theorem 3.1, there exist symmetric, positive definite matrices, $P_i = P_i^T > 0$, and matrices L_i, W_i for both systems (that is, $i = 1, 2$), which satisfy the conditions of the dissipativity lemma,

$$\begin{aligned} P_i A_i + A_i^T P_i &= C_i^T Q_i C_i - L_i^T L_i \\ P_i B_i &= C_i^T (Q_i D_i + N_i) - L_i^T W_i \\ R_i + N_i^T D_i + D_i^T N_i + D_i^T Q_i D_i &= W_i^T W_i \end{aligned}$$

Consider a Lyapunov function, $V(x_1, x_2) = \alpha_1 E_1(x) + \alpha_2 E_2(x)$, where $E_i(x_i) = x_i^T P_i x_i$, correspond to the quadratic energy functions for Σ_1 and Σ_2 . Since $\alpha_1 >$

$0, \alpha_2 > 0$, and the energy functions, $E_i(x_i)$, are positive definite functions, the Lyapunov function, $V(x_1, x_2)$, is a positive definite function of the states of the closed-loop system, x_1, x_2 . The derivative of this Lyapunov function along system trajectories is

$$\frac{d}{dt}V(x_1, x_2) = \alpha_1 \frac{d}{dt}E_1(x_1) + \alpha_2 \frac{d}{dt}E_2(x_2)$$

From the proof of Theorem 3.1, it follows that

$$\frac{d}{dt}E_i(x_i) = -(L_i x_i + W_i f_i)^T (L_i x_i + W_i f_i) + p_i(y_i, f_i)$$

for $i = 1, 2$. Thus,

$$\begin{aligned} \frac{d}{dt}V(x_1, x_2) &= -\alpha_1 (L_1 x_1 + W_1 f_1)^T (L_1 x_1 + W_1 f_1) + \alpha_1 p_1(y_1, f_1) \\ &\quad -\alpha_2 (L_2 x_2 + W_2 f_2)^T (L_2 x_2 + W_2 f_2) + \alpha_2 p_2(y_2, f_2) \end{aligned}$$

Since $\alpha_i > 0$, and $(L_i x_i + W_i f_i)^T (L_i x_i + W_i f_i) \geq 0$, for $i = 1, 2$, it follows that

$$\frac{d}{dt}V(x_1, x_2) \leq \alpha_1 p_1(y_1, f_1) + \alpha_2 p_2(y_2, f_2)$$

Using the conditions for the standard feedback interconnection, that is, $f_2 = y_1$ and $y_2 = -f_1$, gives

$$\frac{d}{dt}V(x_1, x_2) \leq \alpha_1 p_1(y_1, f_1) + \alpha_2 p_2(-f_1, y_1)$$

Since, by hypothesis, $\alpha_1 p_1(y_1, f_1) + \alpha_2 p_2(-f_1, y_1) \leq 0$, the derivative of the Lyapunov function along closed-loop system trajectories is $\frac{d}{dt}V(x_1, x_2) \leq 0$. Thus, by Lyapunov's Second Theorem [27], the closed-loop system is Lyapunov stable.

For asymptotic stability, without loss of generality, assume that Σ_2 is strictly dissipative with respect to the quadratic power function, $p_2(y_2, f_2)$. It is shown that the trivial solution, that is, $x_1 \equiv 0$ and $x_2 \equiv 0$, is the only possible trajectory when $\frac{d}{dt}V(x_1, x_2) \equiv 0$, so that asymptotic stability of the closed-loop system is established using LaSalle's Theorem [30, 27]. Since each of the terms in the expression for the derivative are nonpositive, $\frac{d}{dt}V(x_1, x_2) \equiv 0$, implies that $d_2 = L_2 x_2 + W_2 f_2 = 0$. Since strict dissipativity of Σ_2 implies that the system is observable and transmission zeros are stable, either $f_2 \equiv 0$ or f_2 decreases exponentially with time. Stability of Σ_2 implies that x_2 and y_2 are also exponentially decreasing functions of time. Thus, input and output to the first system f_1 and y_1 are exponentially decreasing, from the requirements of the feedback interconnection. Stability and minimality of the

realization of Σ_1 implies that x_1 is decreasing exponentially. Now, since x_1 and x_2 are exponentially decreasing with time, $\frac{d}{dt}V(x_1, x_2) < 0$, which is a contradiction. Thus, the equilibrium configuration at the origin is the only possible system trajectory with $\frac{d}{dt}V(x_1, x_2) \equiv 0$. Hence, the result. If Σ_1 is strictly dissipative, interchanging the indices of the systems, the argument above again establishes stability of the closed-loop system. \square

Theorem 7.1 is a very powerful result on stability of feedback interconnection of linear time-invariant systems. Next, a series of corollaries are presented, which show that a number of stability results in the literature follow directly by substituting specific power functions and values of scalars to satisfy the sufficient condition in Eq. (7.3). The feedback interconnection of LTI systems, whose stability is characterized in the corollaries, is the standard negative feedback interconnection, shown in Fig. 7.1. The Small Gain Theorem for stability of the feedback interconnection of LTI systems follows by using $\alpha_i = 1$, and power functions for bounded real system, $p_i(y_i, f_i) = f_i^T f_i - y_i^T y_i$, for $i = 1, 2$ in Theorem 7.1.

Corollary 7.1 (Small Gain Theorem) *The feedback interconnection of two bounded real LTI systems (that is, systems satisfying $\|G_i(s)\|_\infty \leq 1$, for $i = 1, 2$) is Lyapunov stable. The closed-loop system is asymptotically stable if either of the systems is strictly bounded real (that is, $\|G_i(s)\|_\infty < 1$, for $i = 1$ or $i = 2$.)*

A more general result for small gain conditions, which is essentially a scaled version of the result above, states that the standard feedback interconnection is stable if the gains $\|G_i(s)\|_\infty \leq \gamma_i$ for $i = 1, 2$, satisfy $\gamma_1 \gamma_2 < 1$. This result follows by using power functions for gain bounded systems, $p_i(y_i, f_i) = \gamma_i^2 f_i^T f_i - y_i^T y_i$, and scalars $\alpha_1 = 1$ and $\alpha_2 = \gamma_1^2$.

Corollary 7.2 (Passivity Theorem) *The feedback interconnection of two positive real systems is Lyapunov stable. If either system is strictly positive real, then the closed-loop system is stable.*

These passivity conditions for stability for the feedback interconnection of passive LTI systems follows by using $\alpha_i = 1$, and power functions for positive real system, $p_i(y_i, f_i) = f_i^T y_i + y_i^T f_i$, for $i = 1, 2$ in Theorem 7.1.

The sector stability results, presented in the next series of corollaries, represent a refinement of the results in the literature[12, 13, 31] since the definition of LTI systems satisfying the sector conditions in a strict sense is weaker than that assumed in the literature.

Corollary 7.3 *The feedback interconnection of an LTI system, Σ_1 , inside sector $[a, b]$, where $a < 0 < b$, with another LTI system, Σ_2 , which is inside sector $[-\frac{1}{b}, -\frac{1}{a}]$, is Lyapunov stable. If either system satisfies its sector constraint in a strict sense, the closed-loop is stable.*

This result follows using $\alpha_1 = 1$, $\alpha_2 = -ab$, and the power functions corresponding to the sector conditions, that is, $p_1(y_1, f_1) = -(y_1 - af_1)^T(y_1 - bf_1)$ and $p_2(y_2, f_2) = -(y_2 + f_2/b)^T(y_2 + f_2/a)$ in Theorem 7.1.

Corollary 7.4 *The feedback interconnection of an LTI system, Σ_1 , inside sector $[a, b]$, where $0 < a < b$, with another LTI system, Σ_2 , which is outside sector $[-\frac{1}{a}, -\frac{1}{b}]$, is Lyapunov stable. If either system satisfies its sector constraint in a strict sense, the closed-loop is stable.*

Using $\alpha_1 = 1$, $\alpha_2 = ab$, and the power functions corresponding to the sector conditions, that is, $p_1(y_1, f_1) = -(y_1 - af_1)^T(y_1 - bf_1)$ and $p_2(y_2, f_2) = (y_2 + f_2/a)^T(y_2 + f_2/b)$ in Theorem 7.1 leads to this result.

Corollary 7.5 *The feedback interconnection of an LTI system, Σ_1 , inside sector $[0, b]$, where $0 < b < \infty$, with another LTI system, Σ_2 , which satisfies $(\Sigma_2 + I/b)$ is positive real, is Lyapunov stable. If either system satisfies its constraints in a strict sense, the closed-loop is stable.*

Using $\alpha_1 = 1$, $\alpha_2 = b$, and the power functions corresponding to the sector conditions, that is, $p_1(y_1, f_1) = -y_1^T(y_1 - bf_1)$ and $p_2(y_2, f_2) = f_2^T(y_2 + f_2/b)$ in Theorem 7.1 leads to this result.

Other combinations of sector conditions for stability of the feedback interconnections can be obtained in a similar fashion. Note that all the results presented thus

far follow by substituting scalar matrices for the coefficient matrices of the quadratic power functions, namely, matrices Q_i, N_i, R_i , for $i = 1, 2$. These results are essentially extensions of the single-input single-output (SISO) stability results to multi-input, multi-output (MIMO) systems. However, assigning general nonscalar values to the matrices Q_i, N_i, R_i , leads to stability which are truly MIMO stability results in that they provide a larger number of parameters to characterize stability characteristics as opposed to system gain (a scalar) or sector conditions (two scalars). For obtaining general results for MIMO systems, consider a matrix oriented expression of the stability condition in Eq. (7.3). Since the equation must be true for all y_1, f_1 , the condition in Eq. (7.3) is equivalent to

$$\begin{bmatrix} \alpha_1 Q_1 + \alpha_2 R_2 & \alpha_1 N_1 - \alpha_2 N_2^T \\ \alpha_1 N_1^T - \alpha_2 N_2 & \alpha_1 R_1 + \alpha_2 Q_2 \end{bmatrix} \leq 0 \quad (7.4)$$

This form of the sufficient condition for stability immediately leads to the following corollary, which is a very significant result itself.

Corollary 7.6 *The feedback interconnection of an LTI system, Σ_1 , which is dissipative with respect to the quadratic power function,*

$$p_1(y_1, f_1) = \alpha_1 \begin{bmatrix} y_1^T & f_1^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y_1 \\ f_1 \end{bmatrix},$$

with another LTI system, Σ_2 , which is dissipative with respect to the quadratic power function

$$p_2(y_2, f_2) = \alpha_2 \begin{bmatrix} y_2^T & f_2^T \end{bmatrix} \begin{bmatrix} -R & N^T \\ N & -Q \end{bmatrix} \begin{bmatrix} y_2 \\ f_2 \end{bmatrix},$$

for any $\alpha_1, \alpha_2 > 0$, is Lyapunov stable. If either system is strictly dissipative with respect to its quadratic power function, then the closed-loop is stable.

Proof: Using the power functions for these dissipative systems in Eq. (7.4) leads to

$$\alpha_1 \alpha_2 \begin{bmatrix} Q - Q & N - N \\ N^T - N^T & R - R \end{bmatrix} = 0 \leq 0$$

Thus, Eq. (7.4) is satisfied, and the result follows from Theorem 7.1. \square

The next three corollaries present new stability results for gain-matrix bounded LTI systems. Since gain-matrix bounded LTI systems form an extension of gain

bounded systems, the following results may be viewed as extension of the small gain conditions.

Corollary 7.7 *The feedback interconnection of an input gain-matrix bounded LTI system with respect to γ_i , with another LTI system which is output gain-matrix bounded with respect to γ_o^{-1} , is Lyapunov stable. If either LTI system is strictly bounded with respect to its gain-matrix, then the closed-loop is stable.*

Proof: Note that the power functions for these systems are $Q_1 = -R_2 = -I$, $N_1 = N_2 = 0$, and $R_1 = -Q_2 = \gamma_i^2$. The result follows substituting these in Eq. (7.4), with $\alpha_1 = \alpha_2 = 1$ and Theorem 7.1. \square

Corollary 7.8 *The feedback interconnection of an output gain-matrix bounded LTI system with respect to γ_o , with another LTI system which is input gain-matrix bounded with respect to γ_i^{-1} , is Lyapunov stable. If either LTI system is strictly bounded with respect to its gain-matrix, then the closed-loop is stable.*

Proof: The power functions for these systems are $Q_2 = -R_1 = -I$, $N_1 = N_2 = 0$, and $R_2 = -Q_1 = \gamma_o^{-2}$. The result follows substituting these in Eq. (7.4), with $\alpha_1 = \alpha_2 = 1$ and Theorem 7.1. \square

Corollary 7.9 *The feedback interconnection of an input-output gain-matrices bounded LTI system with respect to γ_i and γ_o , respectively, with another LTI system which is input-output gain-matrices bounded with respect to γ_o^{-1} and γ_i^{-1} , respectively, is Lyapunov stable. If either LTI system is strictly bounded with respect to its gain matrices, then the closed-loop is stable.*

Proof: Note that the power functions for these systems are $Q_2 = -R_1 = -\gamma_i^2$, $N_1 = N_2 = 0$, and $R_2 = -Q_1 = \gamma_o^{-2}$. The result follows substituting these in Eq. (7.4) with $\alpha_1 = \alpha_2 = 1$ and Theorem 7.1. \square

Numerous other stability results for MIMO systems, like the ones presented above, can be obtained simply by substituting different power functions in the result of Theorem 7.1. This demonstrates that the sufficient conditions for stability of a closed loop system presented in Theorem 7.1 is a very powerful and comprehensive stability result.

Chapter 8

Stability with Feedback Nonlinearities

Application of the framework of dissipative LTI systems is studied in this section for stability of linear time-invariant systems, with memoryless (perhaps time-varying) nonlinearities in negative feedback. LTI systems with feedback nonlinearities are referred to as *Lur  systems* in the literature[11, 30]. Many systems which are primarily characterized as linear time-invariant systems except for a few nonlinear components such as saturating actuators, can be represented as *Lur  systems*. Therefore, traditionally, there has been significant interest in the stability of such systems.

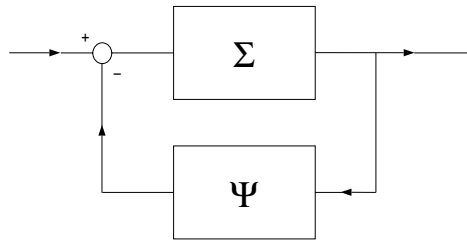


Figure 8.1: LTI System, Σ , with Negative Feedback Nonlinearities.

The stability results are available primarily for sector bounded nonlinearities, with positive real constraints on a transformed linear system [11, 30]. Stability of *Lur  systems* is established in this section for a large class of nonlinearities, with dissipa-

tivity conditions on the LTI system. The results for sector bounded nonlinearities, norm bounded nonlinearities and passive nonlinearities follow as special cases of these results.

Consider an LTI system, Σ , with a minimal state space realization, (A, B, C, D) , such that its dynamics are described by $\dot{x} = Ax + Bf, y = Cx + Df$, where x is an $n \times 1$ vector, and f, y are $m \times 1$ vectors. The feedback nonlinearity is represented by $\Psi(y, t)$, which is memoryless, that is, no dynamics are associated with the feedback loop, as shown in Figure 8.1. Also, the nonlinearity $\Psi(y, t)$ has m outputs with m inputs, $y(t)$. Since negative feedback is assumed, $f = -\Psi(y, t)$. Thus, the time-varying, nonlinear closed-loop system is given by

$$\begin{aligned}\dot{x} &= Ax - B\Psi(y, t) \\ y &= Cx - D\Psi(y, t)\end{aligned}\tag{8.1}$$

Note that the measurement equation for the system above is a nonlinear equation, which is implicit in $y(t)$, output of the linear system. The nonlinearity, $\Psi(y, t)$, is assumed to satisfy smoothness conditions such that the closed-loop system in Eq. (8.1) is well-posed, that is, its solution exists and is unique. Specifically, it is assumed that the nonlinearity, $\Psi(y, t)$ is locally Lipschitz in y and uniformly Lipschitz in t , so that the *Luré system* in Eq. (8.1) is well-posed [11, 30]. Furthermore, it is assumed that $\Psi(0, t) \equiv 0$, for all t , so that the origin is an equilibrium of the closed-loop system.

Theorem 8.1 *If the nonlinearity $\Psi(y, t)$ satisfies*

$$\Psi^T Q \Psi + \Psi^T N y + y^T N^T \Psi + y^T R y \geq 0\tag{8.2}$$

for all t and for all $y \in R^m$, and the LTI system, Σ , is dissipative with respect to the quadratic power function,

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} -R & N^T \\ N & -Q \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

then the origin is a Lyapunov stable equilibrium of the Luré system in Eq. (8.1). If Σ is strictly dissipative with respect to $p(y, f)$, then the origin is a globally asymptotically stable equilibrium.

Proof: Since the LTI system, Σ , is dissipative with respect to the quadratic power function, $p(y, f)$, there exists a symmetric, positive definite matrix, $P = P^T > 0$, such that

$$\begin{aligned} PA + A^T P &= -C^T R C - L^T L \\ PB &= C^T (N^T - R D) - L^T W \\ -Q + N D + D^T N^T - D^T R D &= W^T W \end{aligned}$$

Consider the energy function of the dissipative LTI system, Σ , as a Lyapunov function, $V(x) = x^T P x$, which is a positive definite function, since $P = P^T > 0$. Proceeding in parallel to the proof of Theorem 3.1, the time derivative of the Lyapunov function along the trajectories of the system is

$$\frac{d}{dt}V(x) = -(Lx + Wf)^T(Lx + Wf) + p(y, f)$$

Since $(Lx + Wf)^T(Lx + Wf) \geq 0$, and using the feedback relationship, $f = -\Psi$, shows that

$$\frac{d}{dt}V(x) = -(\Psi^T Q \Psi + \Psi^T N y + y^T N^T \Psi + y^T R y)$$

From the hypothesis, it follows that $\frac{d}{dt}V(x) \leq 0$, so that the origin is a Lyapunov stable equilibrium.

If Σ is strictly dissipative, then global asymptotic stability follows by showing that the trivial trajectory at the origin is the only possible system trajectory when $\frac{d}{dt}V(x) = 0$, by the Invariance Theorems in Ref. [27]. Since $\frac{d}{dt}V(x) = 0$ implies that $d = Lx + Wu = 0$, and $\mathcal{D}(s) = L(sI - A)^{-1}B + W$ has only stable transmission zeros, either input, f , and state, x , are identically zero, or they are exponentially decreasing. However, since $P = P^T > 0$, exponentially decreasing states contradict the condition, $\frac{d}{dt}V(x) = 0$, the trivial state, $x(t) \equiv 0$, is the only possible trajectory. Hence the result. \square

Note that Theorem 8.1 requires that the condition in Eq. (8.2) is satisfied globally, and this leads to global asymptotic stability of the *Luré system* shown in Figure 8.1. However, if the condition in Eq. (8.2) is satisfied in some neighborhood of the origin, but not globally, then the arguments above establish asymptotic stability of the equilibrium at the origin.

The following three corollaries present the results for the special cases, which follow simply by substituting their respective power functions.

Corollary 8.1 *If the nonlinearity $\Psi(y, t)$ satisfies $y^T y - \Psi^T \Psi \leq 0$ for all t and for all $y \in R^m$, and the LTI system, Σ , is bounded real, then the origin is a Lyapunov stable equilibrium of the Luré system in Eq. (8.1). If Σ is strictly bounded real, then the origin is a globally asymptotically stable equilibrium.*

Corollary 8.2 *If the nonlinearity $\Psi(y, t)$ satisfies $y^T \Psi \geq 0$ for all t and for all $y \in R^m$, and the LTI system, Σ , is positive real, then the origin is a Lyapunov stable equilibrium of the Luré system in Eq. (8.1). If Σ is strictly positive real, then the origin is a globally asymptotically stable equilibrium.*

Corollary 8.3 *If the nonlinearity $\Psi(y, t)$ satisfies $(\Psi - ay)^T(\Psi - by) \leq 0$, $a < 0 < b$, for all t and for all $y \in R^m$, and the LTI system, Σ , is inside sector $[-\frac{1}{b}, -\frac{1}{a}]$, then the origin is a Lyapunov stable equilibrium of the Luré system in Eq. (8.1). If Σ satisfies the sector condition in a strict sense, then the origin is a globally asymptotically stable equilibrium.*

Finally, the following result follows by using the power function of a system outside a sector. This result is usually expressed as the absolute stability result[27, 30].

Corollary 8.4 *If the nonlinearity $\Psi(y, t)$ satisfies $(\Psi - ay)^T(\Psi - by) \leq 0$, $0 < a < b$, for all t and for all $y \in R^m$, and the LTI system, Σ , is outside sector $[-\frac{1}{a}, -\frac{1}{b}]$, then the origin is a Lyapunov stable equilibrium of the Luré system in Eq. (8.1). If Σ satisfies the sector condition in a strict sense, then the origin is a globally asymptotically stable equilibrium.*

For a SISO system, the result above corresponds to the Circle Criterion [30], since a SISO LTI system being outside sector $[-\frac{1}{a}, -\frac{1}{b}]$ implies that the frequency response of the system lies outside the circle intersecting the real axis at $-\frac{1}{a}$ and $-\frac{1}{b}$ in the frequency plane.

Finally, the framework of dissipative LTI systems extends the results for stability of *Luré systems*, beyond unification of the SISO results, to MIMO stability results, which may be obtained by substituting their specific power functions in the stability result of Theorem 8.1.

Chapter 9

Quadratic Stability

Quadratic stability deals with stability of uncertain linear systems for all time-varying parameter variations within a predefined uncertainty set.

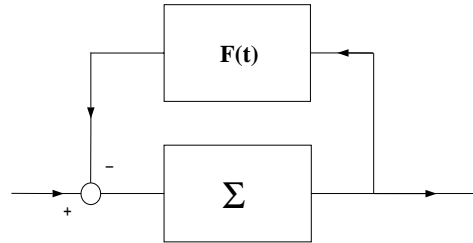


Figure 9.1: Uncertainty Configuration for Quadratic Stability.

The time-varying linear systems, whose stability is being investigated, are given as

$$\dot{x}(t) = (A + \Delta A(t))x(t) \quad (9.1)$$

where $x(t)$ is an $n \times 1$ state space vector, A is the constant, known part of the system matrix, and $\Delta A(t)$ is the unknown, time-varying part of the system matrix. The unknown time-varying component of the system matrix is assumed to be within an uncertainty set, that is, $\Delta A(t) \in \mathbf{\Delta}$, where $\mathbf{\Delta}$ is a known, predefined uncertainty set. The uncertain linear system in Eq. (9.1) is said to be *quadratically stable* if there exists a parameter-independent quadratic Lyapunov function which guarantees stability for

all time-varying parametric variations within the uncertainty set. Quadratic stability is contrasted with a weaker concept of robust stability, where the uncertain parameters in ΔA are constant, but unknown. Robust stability may be established by ensuring that the system eigenvalues are in the open, left-half plane for all uncertain values of the parameters within the uncertainty set. A discussion of the distinction between quadratic stability and robust stability can be found in Ref. [32].

The uncertain component of the system matrix, $\Delta A(t)$, may depend on a smaller set of parameters, represented by an $m \times m$ matrix, $F(t)$, such that $\Delta A(t) = -BF(t)C$, where B is a $n \times m$ matrix, and C is a $m \times n$ matrix. The matrices B, C describe the distribution of the uncertain parameters in the matrix, $F(t)$, onto the uncertain component, $\Delta A(t)$. The uncertain parameter matrix, $F(t)$, belongs to a parametric uncertainty set, Δ_F , so that $F(t) \in \Delta_F$ implies that $\Delta A(t) \in \Delta$. Thus, the uncertain linear system can be represented as shown in Fig. 9.1, as a known linear system with state space realization, $(A, B, C, 0)$, with the matrix of uncertain parameters, $F(t)$, in negative feedback. The minus sign in the description of system matrix uncertainty had been selected to be consistent with the negative feedback paradigm. A general approach to representing uncertain linear systems as a known LTI system, with the parametric uncertainty in a negative feedback matrix, is referred to as "pulling out the Δ 's" in the literature[24, 22]. In general, the direct feed through matrix (the D matrix) of the linear system may not be zero, as in the simpler description, above. In this general case, the uncertain component of the system matrix is expressed as $\Delta A(t) = -BF(t)(I + DF(t))^{-1}C$.

Quadratic stability of uncertain linear systems has primarily been studied for norm bounded parametric uncertainty, that is, for systems where the parametric uncertainty set Δ_F is defined in terms of norm bounds. The next result develops the extension of quadratic stability results to problems where the known linear, time-invariant part of the uncertain system is dissipative.

Theorem 9.1 *Consider an uncertain system, $\Sigma : \dot{x}(t) = (A + \Delta A(t))x(t)$, where $\Delta A(t) = -BF(t)(I + DF(t))^{-1}C$, and $F(t)$ is a matrix of uncertain, time-varying parameters, which satisfy*

$$Q - NF(t) - F^T(t)N^T + F^T(t)RF(t) \leq 0 \quad (9.2)$$

If (A, B, C, D) is a minimal realization of an LTI system, which is dissipative with

respect to the quadratic power function

$$p(y, f) = \begin{bmatrix} y^T & f^T \end{bmatrix} \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ f \end{bmatrix}$$

then the origin is a Lyapunov stable equilibrium of the uncertain system, Σ . If the LTI system is strictly dissipative, then the uncertain system is quadratically stable.

Proof: Since the LTI system is dissipative with respect to the quadratic power function, $p(y, f)$, there exists a symmetric, positive definite matrix, $P = P^T > 0$, and matrices L and W which satisfy the dissipativity lemma,

$$\begin{aligned} PA + A^T P &= C^T Q C - L^T L \\ PB &= C^T (QD + N) - L^T W \\ R + N^T D + D^T N + D^T Q D &= W^T W \end{aligned} \quad (9.3)$$

Consider the energy function of this dissipative LTI system as a Lyapunov function for the uncertain linear system, $V(x) = x^T P x$, where $P = P^T > 0$ is a symmetric, positive definite matrix which satisfies the dissipativity lemma above. The time derivative of this Lyapunov function along the trajectories of the uncertain linear system is

$$\begin{aligned} \frac{d}{dt} V(x) &= x^T (PA + A^T P) x - x^T P B F(t) (I + D F(t))^{-1} C x \\ &\quad - x^T C^T (I + D F(t))^{-T} F^T(t) B^T P x \end{aligned} \quad (9.4)$$

Let $H(t) = (I + D F(t))^{-1}$, and drop the argument t from $H(t)$ and $F(t)$ to simplify the following expressions. Using the first two relations in Eq. (9.3) gives

$$\begin{aligned} \frac{d}{dt} V(x) &= x^T (C^T Q C - L^T L) x - x^T (C^T (QD + N) - L^T W) F H x \\ &\quad - x^T H^T F^T ((QD + N)^T C - W^T L) x \\ &= -x^T L^T L x + x^T L^T W F H x + x^T H^T F^T W^T L x + x^T C^T Q C x \\ &\quad - x^T C^T (QD + N) F H x - x^T H^T F^T (QD + N)^T C x \end{aligned} \quad (9.5)$$

Adding and subtracting $x^T H^T F^T W^T W F H x$ to Eq. (9.5) for completing the square, and using the last relationship in Eq. (9.3), leads to

$$\begin{aligned} \frac{d}{dt} V(x) &= -x^T L^T L x + x^T L^T W F H x + x^T H^T F^T W^T L x - x^T H^T F^T W^T W F H x \\ &\quad + x^T C^T Q C x - x^T C^T (QD + N) F H x - x^T H^T F^T (QD + N)^T C x \\ &\quad + x^T H^T F^T (R + N^T D + D^T N + D Q D) F H x \end{aligned}$$

Collecting terms and simplifying

$$\begin{aligned} \frac{d}{dt}V(x) = & -x^T(L - WFH)^T(L - WFH)x + x^T(C - DFH)^TQ(C - DFH)x \\ & -x^T(C - DFH)^TNFHx - x^TH^TF^TN^T(C - DFH)x \\ & +x^TH^TF^TRFHx \end{aligned} \quad (9.6)$$

Note the identity, $H = (C - DFH)$, which may be verified readily from the definition of H . Using this identity in Eq. (9.6) gives

$$\begin{aligned} \frac{d}{dt}V(x) = & -x^T(L - WFH)^T(L - WFH)x \\ & +x^TH^T(Q - NF - F^TN^T + F^TRF)Hx \end{aligned} \quad (9.7)$$

Since $x^T(L - WFH)^T(L - WFH)x \geq 0$, and by the hypothesis, the second term in Eq. (9.7) is nonpositive, it follows that $\frac{d}{dt}V(x) \leq 0$. Thus, the uncertain, linear, time-varying system is Lyapunov stable about the origin.

If the LTI system $G(s) = C(sI - A)^{-1}B + D$ is strictly dissipative with respect to the quadratic power function, $p(y, f)$, then $\mathcal{D}(s) = L(sI - A)^{-1}B + W$ is a minimal realization of a stable, minimum phase transfer function. Denote $f = -FHx$, so that $y' \equiv (L - WFH)x = Lx + Wf$. Thus, from Eq. (9.7) it follows that $\frac{d}{dt}V(x) = 0$ implies $y' = 0$. Since $\mathcal{D}(s)$ is minimum phase, $y' = 0$ implies that either $x \equiv 0$, or that $f = -FHx$ and x are exponentially decreasing functions of time. The later possibility leads to a contradiction, since it implies $\frac{d}{dt}V(x) < 0$. Hence, $x \equiv 0$ is the only possible trajectory when $\frac{d}{dt}V(x) = 0$. Therefore, the uncertain system, Σ , is asymptotically stable for all $F(t) \in \Delta_F$ by the Invariance Theorems in Ref. [27], that is, the uncertain linear system, Σ , is quadratically stable. \square

The stability result of Theorem 9.1 represents a general conditions for quadratic stability of a large class of uncertain linear systems. Particular results when parametric variations belong to uncertainty sets characterized as in Eq. (9.2) follow simply by substituting the corresponding power functions in Theorem 9.1. Specifically, the case for norm bounded parametric uncertainty is presented in the following corollary, since this case has been studied extensively in the literature.

Corollary 9.1 *Consider an uncertain system, $\Sigma : \dot{x}(t) = (A + \Delta A(t))x(t)$, where $\Delta A(t) = -BF(t)(I + DF(t))^{-1}C$, and $F(t)$ is a matrix of uncertain, time-varying*

parameters, is norm bounded, that is it satisfies

$$I - F^T(t)F(t) \geq 0 \quad (9.8)$$

for all t . If (A, B, C, D) is a minimal realization of an LTI system, which is bounded real, then the origin is a Lyapunov stable equilibrium of the uncertain system, Σ . If the LTI system is strictly bounded real, then the uncertain system is quadratically stable.

This corollary follows from Theorem 9.1 by substituting $Q = -I$, $R = I$, and $N = 0$. Using the Riccati equation characterization for strictly bounded real systems, the corollary above states the quadratic stability result in Ref. [32]. Thus, corollary 9.1 presents the Riccati equation conditions for quadratic stability with norm bounded uncertainty[32], in terms of bounded real systems, which has a more general LMI characterization.

Similar results can be obtained for quadratic stability when the known LTI system is positive real, by substituting $N = I$ and $Q = R = 0$, and for sector bounded LTI systems, by substituting $Q = abI$, $N = \alpha I$, and $R = I$, where $\alpha = (a + b)/2$. These results are not stated as separate corollaries to avoid repetition. Thus, it is seen that the framework of dissipative LTI systems presents a general framework for quadratic stability of a large class of uncertain linear systems.

Chapter 10

Selection of Quadratic Power Functions

Characterization of stable LTI systems in terms of dissipativity with respect to certain quadratic power functions is necessary for the application of the stability results developed in the three previous sections. The selection of quadratic power functions, such that a given LTI system is dissipative with respect to the power function, is addressed in this section. This problem is posed as optimization of linear objective functions with LMI constraints, or positive semidefinite programming problems, which can be solved using efficient convex programming techniques [19, 33, 34].

The LMI characterization of dissipative LTI systems, given by Theorem 3.1, forms the foundation for the techniques presented in this section. Let a minimal realization of a given stable LTI system be (A, B, C, D) , and the coefficient matrices for a given quadratic power function be $Q = Q^T, R = R^T$, and N . First, consider the problem of determining whether this LTI system is dissipative with respect to the quadratic power function. An LMI approach to this problem, based on the result of Theorem 3.1 is discussed here. This feasibility problem is posed as maximization of a linear objective function, $J = t$, with respect to a scalar variable, t , and a symmetric matrix, $P = P^T$, such that (1) $P - tI \geq 0$ and

$$(2) \begin{bmatrix} C^T Q C - P A - A^T P & C^T (Q D + N) - P B \\ (Q D + N)^T C - B^T P & R + N^T D + D^T N + D^T Q D \end{bmatrix} \geq 0$$

If the parameter t attains a positive value, then the first condition implies that the

symmetric matrix $P = P^T$ is positive definite, and the second condition implies that it satisfies the dissipativity LMI, so that the LTI system is dissipative. Otherwise, the system is not dissipative with respect to the given quadratic power function. Note that the problem of maximization of a linear objective function subject to LMI constraints is a convex programming problem. Thus, a global maximum for the parameter t exists, and may be computed using efficient numerical techniques [14, 34].

Often, it is desirable to obtain the coefficient matrices, $Q = Q^T, R = R^T$, and N of the power function, such that a given LTI system is dissipative with respect to that power function. Note that the matrix inequality of the dissipativity lemma is linear with respect to the matrix $P = P^T$ as well as the coefficient matrices, $Q = Q^T, R = R^T$, and N . Therefore, the problem of selecting quadratic power functions can be posed as an LMI problem, with the coefficient matrices also being considered as optimization variables. However, since every stable LTI is dissipative with respect to numerous quadratic power functions, it is desirable to determine power functions which somehow provide a tight characterization of the LTI system under consideration. The motivation for tight characterization of plants is to obtain a larger class of controllers that stabilize the closed-loop system, using the stability result of Theorem 7.1; or, to enlarge the uncertainty sets described in Theorems 8.1 and 9.1. A number of approaches to such selection of power functions are discussed in the rest of this section.

Consider the power function corresponding to the \mathcal{H}_∞ norm of a stable LTI system, Σ , whose transfer function is $G(s) = C(sI - A)^{-1}B + D$, and (A, B, C, D) is a minimal realization. From Section 2, it follows that if a parameter, γ , is larger than the \mathcal{H}_∞ norm of Σ , that is, if $\|G(s)\|_\infty \leq \gamma$, then the system is dissipative with respect to a quadratic power function with coefficient matrices, $Q = -I, R = \gamma^2 I$, and $N = 0$. Thus, to select a tight power function of this form, the parameter γ must be minimized until it attains its minimum value, which is $\|G(s)\|_\infty$. However, since γ^2 appears in the dissipativity LMI, rather than γ , the objective function $J = \gamma^2$ leads to a positive semidefinite programming problem. Thus, selection of quadratic power functions with the structure of system gain is accomplished by minimizing a linear objective function, $J = \gamma^2$, with respect to a symmetric, positive definite matrix, $P = P^T > 0$, and positive scalar, γ^2 under the LMI constraint for bounded realness, that is,

$$\begin{bmatrix} PA + A^T P + C^T C & C^T D + P B \\ D^T C + B^T P & D^T D - \gamma^2 I \end{bmatrix} \leq 0$$

By definition, the optimum value is $J = \|G(s)\|_\infty^2$. Thus, this approach also provides a convex programming approach to computing the \mathcal{H}_∞ norm of a stable LTI system. It is noted here that this approach to computing \mathcal{H}_∞ norm of a stable LTI system is very robust and efficient, since it involves positive semidefinite programming for the minimization of a linear objective subject to LMI constraints. This is in contrast to computing the \mathcal{H}_∞ norm using a bisection technique along with checking imaginary axis eigenvalues of a Hamiltonian matrix, which becomes an ill-conditioned problem as the parameter γ gets closer to the norm. For large dimension systems, this positive semidefinite programming approach is computationally intensive, but provides accurate results.

Next, an LMI approach is presented to obtain a tight characterization of an LTI system in terms of input gain-matrix boundedness. The approach is to minimize the size of the input gain-matrix, as measured by its Frobenius norm, with respect to which the system is input gain-matrix bounded. Note that for the special case of gain bounded systems, with scalar input gain-matrices, the Frobenius norm of the scalar input gain-matrix is proportional to the gain of the system. Thus, for this special case, minimizing the Frobenius norm of the input gain-matrix is equivalent to minimizing the gain of the system. For the general case, the approach for obtaining a tight characterization of an LTI system in terms of input gain-matrix boundedness is to determine the minimum Frobenius norm, input gain-matrix. Recall that coefficient matrices for the quadratic power functions of input gain-matrix bounded systems are $Q = -I$, $N = 0$, and $R = \gamma^2 I$. The square of the Frobenius norm of γ is the trace of the symmetric positive definite matrix, $R = R^T > 0$. The problem is reduced to minimizing the trace of a positive definite matrix, $R = R^T > 0$, (with $Q = -I$ and $N = 0$ remaining fixed), while satisfying the dissipativity LMI. A minimum Frobenius norm input gain-matrix, γ , is given by the square root of R . Thus, the LMI problem is to minimize a linear objective function, $J = \text{Trace } R$, with respect to positive definite matrices, $P = P^T > 0$ and $R = R^T > 0$, under the LMI constraint,

$$\begin{bmatrix} PA + A^T P + C^T C & C^T D + PB \\ D^T C + B^T P & D^T D - R \end{bmatrix} \leq 0$$

An input gain-matrix bounding the given LTI system with minimum Frobenius norm is given by $\gamma = R^{1/2}$.

A similar approach can be used to determine an output gain-matrix bounding an LTI system, $\gamma_o = \gamma_o^T > 0$, such that Frobenius norm of its inverse is maximized. From

section 2, the coefficient matrices for the quadratic functions of output gain-matrix bounded systems are $R = I, N = 0$, and $Q = -\frac{1}{\sigma^2}$. Let $V = \frac{1}{\sigma^2}$, so that the trace of this symmetric positive definite matrix, $\text{Trace } V$, is the square of the Frobenius norm of $\frac{1}{\sigma}$. The LMI problem becomes to maximize a linear objective function, $J = \text{Trace } V$, with respect to symmetric, positive definite matrices, $P = P^T > 0$, and $V = V^T > 0$, such that

$$\begin{bmatrix} PA + A^T P + C^T V C & C^T V D + P B \\ D^T V C + B^T P & D^T V D - I \end{bmatrix} \leq 0$$

An output gain matrix, bounding the LTI system, is given by $\frac{1}{\sigma} = V^{-1/2}$.

An LMI approach to determine parameters, a, b , such that $a < 0 < b$ and a given LTI system lies inside sector $[a, b]$ is presented next. The characterization of tightness of sector boundedness is motivated by the frequency-domain interpretation of a sector bounded single-input single-output (SISO) LTI system in the frequency plane. A SISO LTI system inside sector $[a, b]$ has its frequency response within a circle in the frequency plane, which intersects the real axis at a and b , with the center of this circle being on the real axis. The center of this circle is at $(a + b)/2$ and the radius of this circle is equal to $(b - a)/2$. When $a = -\gamma$ and $b = \gamma$, this corresponds to the \mathcal{H}_∞ gain of the system being bounded by γ , which implies that its frequency response lies within a circle centered at the origin, having radius γ . Minimizing γ^2 , therefore, corresponds to minimizing the square of the radius of a circle centered at the origin (which is proportional to the area of the circle), such that the frequency response lies within the circle. Sector boundedness allows the center of this circle to lie anywhere on the real axis of the frequency plane. Thus, a tight characterization of a SISO LTI system in terms of sector boundedness is to determine the smallest circle centered on the real axis, such that the frequency response of the LTI system lies within the circle. The smallness of the circle is measured in terms of the area of the circle, which is proportional to square of the radius of this circle, that is, $(b - a)^2/4$.

With motivation from the frequency-domain interpretation above, the optimization problem is to select the parameters a and b such that $(b - a)^2/4$ is minimized, with the LTI system being inside sector $[a, b]$. Note that $(b - a)^2/4 = \alpha^2 + \delta_1$, where $\alpha = (a + b)/2$, and $\delta_1 = -ab > 0$. Furthermore, to obtain a linear objective, set $J = \delta_1 + \delta_2$, with $\delta_2 \geq \alpha^2$. These manipulations are performed to formulate the optimization problem of selecting the parameters a, b as the following LMI problem: To minimize the linear objective function, $J = \delta_1 + \delta_2$, with respect to a symmetric

positive definite matrix, $P = P^T > 0$, and positive scalars $\delta_1, \delta_2, \alpha$, under the following LMI constraints:

$$\begin{bmatrix} PA + A^T P + C^T C & PB - C^T(\alpha I - D) \\ B^T P - (\alpha I - D)^T C & \delta_1 I - \alpha(D + D^T) + D^T D \end{bmatrix} \leq 0$$

and

$$\begin{bmatrix} \delta_2 & \alpha \\ \alpha & 1 \end{bmatrix} \geq 0$$

The optimal values of a and b are obtained as $a = \alpha - \sqrt{\alpha^2 + \delta_1}$ and $b = \alpha + \sqrt{\alpha^2 + \delta_1}$, from the optimal values for α and δ_1 .

Another tight characterization of LTI systems can be obtained in terms of symmetric positive definite matrices, $R = R^T > 0$, and general nonzero matrix, N , with $Q = -I$, as described by the following LMI problem. Minimize a linear objective function, $J = \text{Trace } R + \delta_1$, with respect to symmetric positive definite matrices, $R = R^T > 0, P = P^T > 0$, and a general matrix, N , which satisfy the LMI constraints,

$$\begin{bmatrix} PA + A^T P + C^T C & C^T(D - N) + PB \\ (D - N)^T C + B^T P & D^T D - N^T D - D^T N - R \end{bmatrix} \leq 0$$

and

$$\begin{bmatrix} \delta_1 I & N^T \\ N & I \end{bmatrix} \geq 0$$

A similar LMI optimization could be performed with $R = I$, arbitrary nonzero matrix, N , and $Q = Q^T < 0$.

Once power functions providing a tight characterization of uncertain plants to be controlled have been obtained, controllers are synthesized to enhance system performance while ensuring that they satisfy the dissipativity criteria for robust stability. Synthesis of robustly stabilizing dissipative controllers which enhance overall performance of the closed-loop system is an open research area, which will be explored in the future.

Chapter 11

Spring-Mass-Damper Example

This section demonstrates the application of the results developed in this report for robust controller synthesis using a spring-mass-damper system. First, characterization of this LTI system in terms of various power functions is presented. The later part of this section discusses an approach for synthesis of linear, quadratic Gaussian (LQG) controllers which satisfy stability criteria for dissipative systems.

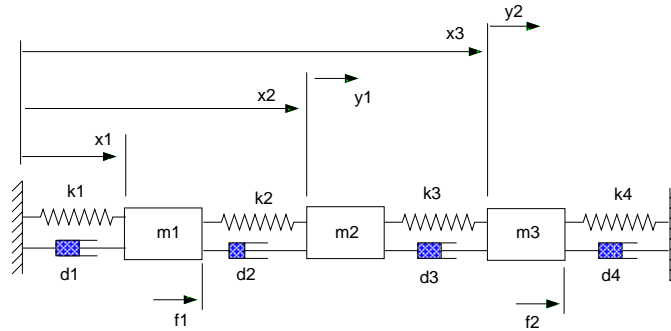


Figure 11.1: Three Spring-Mass-Damper System.

The system used for the numerical example consists of three masses interconnected by springs and dampers as shown in Figure 11.1. Values used for the masses, spring constants, and damping coefficients are shown in Table 11.1. Input forces are applied at masses 1 and 3, and velocities of masses 2 and 3 are measured. Equations of motion for this system are developed using a Lagrangian approach, as described in

i	m_i	d_i	k_i
1	2.0	0.5	5.0
2	1.0	0.2	2.0
3	2.0	0.15	2.0
4		0.45	5.0

Table 11.1: Parameters of Three Spring-Mass-Damper System.

i	Open-loop Eigenvalue	ω_i (rad/sec)	ρ_i
1	$-0.0812 \pm 1.3145j$	1.3170	0.0616
2	$-0.1625 \pm 1.8643j$	1.8713	0.0868
3	$-0.2563 \pm 2.3867j$	2.4005	0.1068

Table 11.2: Natural Frequencies and Damping Ratios for Vibration Modes.

the Introduction. Natural frequencies and damping ratios for three modes of vibration of this system are given in Table 11.2. Six states for a minimal realization of these dynamics as a state space model are positions of the three masses followed by their velocities. This state space realization is given in Table 11.3. Thus, the model used is a multi-input, multi-output (MIMO) system, with noncollocated sensors and actuators. The singular value plot for this system is shown in Figure 11.2.

Before proceeding with the examples for the MIMO system, some examples are presented for a single-input, single-output (SISO) system, corresponding to input force applied at mass 1 with velocity of mass 3 being measured. This is done because the results for SISO systems can be visualized, in terms of the frequency response of the system, and circles in the frequency plane exhibiting the frequency domain conditions for dissipative LTI systems.

Bode magnitude plot for this system are shown in Figure 11.3, and its Nyquist plot is shown in Figure 11.4. \mathcal{H}_∞ norm of the system is computed to be 1.516. Using a gain bound of $\gamma = 1.75$, it follows that this system is dissipative with respect to a quadratic power function, $Q = -1$, $R = 3.0625$, and $N = 0$. Using convex programming techniques and software from Refs. [33, 34] to solve the feasibility

$$A = \begin{bmatrix} 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \\ -3.50 & 1.00 & 0.00 & -0.35 & 0.10 & 0.00 \\ 2.00 & -4.00 & 2.00 & 0.20 & -0.35 & 0.15 \\ 0.00 & 1.00 & -3.50 & 0.00 & 0.075 & -0.30 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \\ 0.5 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.5 \end{bmatrix} \quad C = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{bmatrix} \quad D = \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}$$

Table 11.3: State Space Realization for Spring-Mass-Damper Model.

LMI of the previous section, a symmetric, positive definite matrix which satisfies the dissipativity LMI, Eq. (3.2), with the values above for Q , N , and R , is computed to be

$$P = \begin{bmatrix} 57.180 & -37.370 & 22.555 & -1.901 & -6.451 & -0.099 \\ -37.370 & 60.795 & -49.074 & 3.984 & 2.387 & -3.491 \\ 22.555 & -49.074 & 83.735 & 0.353 & 1.775 & 2.463 \\ -1.901 & 3.984 & 0.353 & 13.335 & -4.157 & 4.279 \\ -6.451 & 2.387 & 1.775 & -4.157 & 13.752 & -8.123 \\ -0.099 & -3.491 & 2.463 & 4.279 & -8.123 & 21.873 \end{bmatrix}$$

All eigenvalues of this matrix P are positive, with the minimum eigenvalue being 7.024. All eigenvalues of the symmetric matrix on the left-hand side of the dissipativity LMI in Eq. (3.2) are nonpositive for this symmetric matrix, P , thus showing that the LMI is satisfied. Note that since \hat{R} used in section 4 is nonsingular, the same matrix, P , also satisfies the algebraic Riccati inequality of Eq. (4.1). The dashed circle in Figure 11.3 is a circle of radius 1.75 centered at the origin. The frequency response of this system is seen to lie within the dashed circle, as expected from the frequency domain conditions of section 5.

Next, a minimum γ satisfying the gain boundedness LMI is computed for the SISO system, using the LMI approach presented in the previous section. Using convex programming software [33, 34] for the positive semidefinite program of the previous section, the minimum value of γ is computed to be $\gamma = 1.516$, which is the \mathcal{H}_∞ norm of the system. A symmetric positive definite matrix satisfying the gain boundedness LMI for this value of γ is computed as

$$P = \begin{bmatrix} 7.804 & -0.319 & 1.590 & 0.153 & 0.022 & -0.175 \\ -0.319 & 5.296 & -0.399 & -0.169 & -0.184 & -0.638 \\ 1.590 & -0.399 & 8.429 & 0.404 & 0.650 & 0.458 \\ 0.153 & -0.169 & 0.404 & 2.869 & 0.853 & 0.863 \\ 0.022 & -0.184 & 0.650 & 0.853 & 1.759 & 0.901 \\ -0.175 & -0.638 & 0.458 & 0.863 & 0.901 & 3.016 \end{bmatrix}$$

All eigenvalues of the matrix, P , are positive, with the minimum eigenvalue being 1.1570, and all eigenvalues of the symmetric matrix on the left-hand side of the gain boundedness LMI are nonpositive. The dotted circle in Figure 11.3 is centered at the origin with a radius of 1.516, and it can be seen that this is the smallest circle centered at the origin such that the frequency response of the system remains within the circle.

Finally, using the LMI approach of the previous section, tight parameters a and b are determined corresponding to the smallest circle that is centered on the real axis, and contains the frequency response of the system. The optimization results in $a = -0.488$ and $b = 1.525$ as the tightest parameters such that the LTI system lies inside sector $[a, b]$. Therefore, the system is dissipative with respect to a quadratic power function with $Q = -1$, $R = 0.744$, and $N = 0.519$. A symmetric positive definite matrix satisfying the sector boundedness lemma for $a = -0.488$ and $b = 1.525$, that is, the dissipativity LMI with values of Q , N , and R as above, is

$$P = \begin{bmatrix} 8.001 & -3.884 & 4.739 & -0.092 & -0.324 & 0.413 \\ -3.884 & 7.736 & -4.066 & 0.320 & -0.027 & -0.593 \\ 4.739 & -4.066 & 8.417 & -0.334 & 0.498 & 0.293 \\ -0.092 & 0.320 & -0.334 & 2.168 & -0.015 & 1.346 \\ -0.324 & -0.027 & 0.498 & -0.015 & 1.903 & -0.088 \\ 0.413 & -0.593 & 0.293 & 1.346 & -0.088 & 2.296 \end{bmatrix}$$

The minimum eigenvalue of the matrix P is 0.819, and it can be verified that all eigenvalues of the matrix on the right-hand side of the sector boundedness LMI are

nonpositive. Figure 11.5 shows the Nyquist plot again. The dashed circle is centered on the real axis, and intersects the real axis at $a = -0.488$ and $b = 1.525$. It is seen that this is the smallest circle centered on the real axis such that the frequency response of the LTI system remains within the circle. For comparison, the dotted circle, of radius $\gamma = 1.516$ centered at the origin, corresponding to the \mathcal{H}_∞ norm of the system, is also shown in Figure 11.5. It is obvious from Figure 11.5 that sector boundedness provides a tighter characterization of this system as opposed to the \mathcal{H}_∞ norm characterization. Thus, this example demonstrates that the use of more general quadratic power functions, than those for gain-boundedness, can lead to a tighter characterization of the plant for robust stabilization.

The two-input, two-output model of the spring-mass-damper system is be used for the following numerical examples. First, the \mathcal{H}_∞ norm of the system is calculated using the LMI approach presented in the previous section, that is, by minimizing the gain bound, γ , under the constraint that it satisfies the gain boundedness lemma. The minimum value is computed to be $\gamma = 2.4788$, which indeed is the \mathcal{H}_∞ norm of the system. Thus, one tight characterization of the LTI system is that it is dissipative with respect to a quadratic power function with coefficients $Q = -I$, $N = 0$, and $R = 6.1447I$, where the identity matrix I is of order 2. A symmetric positive definite matrix satisfying the dissipativity LMI for this power function is

$$P = \begin{bmatrix} 10.666 & 0.145 & 0.919 & 0.651 & 0.595 & -0.029 \\ 0.145 & 7.289 & -1.839 & -0.727 & -0.058 & 0.232 \\ 0.919 & -1.839 & 14.737 & -0.077 & -0.304 & 0.009 \\ 0.651 & -0.727 & -0.077 & 3.711 & 1.295 & 0.956 \\ 0.595 & -0.058 & -0.304 & 1.295 & 2.348 & 0.893 \\ -0.029 & 0.232 & 0.009 & 0.956 & 0.893 & 4.646 \end{bmatrix}$$

The minimum eigenvalue of this matrix, P , is 1.518, and the LMI for gain boundedness is satisfied.

Next, tight sector bounds for this MIMO system are computed using the LMI approach as above. The rationale for this optimization is that the process minimizes the sum of areas of the circle in frequency plane for each channel. In practice, the same approach is used as for selecting tight sector bounds for SISO systems. The solution to the LMI problem gives the optimal parameters as $a = -0.582$ and $b = 2.662$. Unfortunately, there is no simple way to visualize this result, and its verification follows simply by noting that the sector boundedness lemma with parameters $a =$

-0.582 and $b = 2.662$ is satisfied by the following symmetric positive definite matrix,

$$P = \begin{bmatrix} 10.784 & -4.256 & 4.984 & 0.087 & 0.091 & 0.589 \\ -4.256 & 9.044 & -4.631 & 0.083 & -0.041 & -0.275 \\ 4.984 & -4.631 & 11.340 & -0.584 & 0.064 & 0.112 \\ 0.087 & 0.083 & -0.584 & 3.164 & 0.075 & 1.458 \\ 0.091 & -0.041 & 0.064 & 0.075 & 2.276 & 0.022 \\ 0.589 & -0.275 & 0.112 & 1.458 & 0.022 & 3.170 \end{bmatrix}$$

It can be verified that this matrix is positive definite, and its smallest eigenvalue is 1.680. Note that there the system is not positive real, so no results can be computed for the positive realness lemma.

Next the computation of a minimum Frobenius norm input gain matrix is performed, such that the LTI system is input matrix gain bounded, as discussed in section 2. This computation is implemented as a linear objective with LMI constraints, presented in the previous section. The optimal value of a input matrix gain bound is $\gamma_i = \begin{bmatrix} 1.771 & 0.697 \\ 0.697 & 1.820 \end{bmatrix}$. Equivalently, the LTI system is dissipative with respect to a quadratic power function with coefficient matrices $Q = -I$, $N = 0$, and $R = \begin{bmatrix} 3.624 & 2.504 \\ 2.504 & 3.799 \end{bmatrix}$. The dissipativity LMI is satisfied for these coefficients of the quadratic power function by the symmetric, positive definite matrix,

$$P = \begin{bmatrix} 11.256 & -2.987 & 6.073 & 0.397 & 0.581 & -0.112 \\ -2.987 & 9.975 & -3.333 & -0.734 & 0.002 & 0.413 \\ 6.073 & -3.333 & 12.132 & 0.313 & -0.471 & -0.194 \\ 0.397 & -0.734 & 0.313 & 3.699 & 0.655 & 2.187 \\ 0.581 & 0.002 & -0.471 & 0.655 & 2.893 & 0.625 \\ -0.112 & 0.413 & -0.194 & 2.187 & 0.625 & 3.900 \end{bmatrix}$$

The smallest eigenvalue of this matrix is 1.529, demonstrating that it is positive definite; and it can be verified that the eigenvalues of the right-hand side of the LMI are nonpositive.

Similarly, following the approach presented in the previous section, an output gain matrix bound is computed as $\gamma_o = \begin{bmatrix} 2.379 & 1.535 \\ 1.535 & 2.479 \end{bmatrix}$. The LTI system is dissipative

with respect to a quadratic power function with coefficients $Q = -\begin{bmatrix} 0.678 & -0.595 \\ -0.595 & 0.640 \end{bmatrix}$,

$N = 0$, and $R = I$, satisfying the dissipativity LMI with

$$P = \begin{bmatrix} 5.073 & -3.005 & 1.940 & 0.262 & 0.344 & -1.108 \\ -3.005 & 4.592 & -3.407 & -0.726 & 0.002 & 0.575 \\ 1.940 & -3.407 & 6.119 & 1.181 & -0.313 & -0.121 \\ 0.262 & -0.726 & 1.181 & 1.265 & -0.353 & 0.337 \\ 0.344 & 0.002 & -0.313 & -0.353 & 0.973 & -0.396 \\ -1.108 & 0.575 & -0.121 & 0.337 & -0.396 & 1.543 \end{bmatrix}$$

The minimum eigenvalue of P is 0.680.

Many other quadratic power functions can be obtained such that the given system is dissipative with respect to that power function. The quadratic power functions computed in this section for the same system demonstrate that a system is dissipative with respect to many quadratic power functions.

Once the plant has been characterized in terms of dissipativity with respect to a quadratic power function, synthesis of a controller that is dissipative with respect to another power function which satisfies the sufficient condition for stability is performed for stability robustness. Various approaches for such robust controller synthesis are possible. \mathcal{H}_∞ control theory provides a framework for synthesis of robust controllers for gain bounded systems, and synthesis of positive real controllers is discussed in Refs [15, 35]. Extension of these techniques to general dissipative systems is being pursued currently. An approach to design MIMO controllers employing optimal linear regulators and state estimators such that the overall controller satisfies the stability criteria is discussed in the next.

Full state feedback is assumed for the design of the feedback gain matrix which optimizes a quadratic performance index using the linear regulator theory. Linear state estimators are designed such that the overall controller satisfies dissipativity requirements for robust stability. The approach for synthesis of state estimators is that of design of optimal Kalman filters, except that the noise covariance matrices are design parameters rather than a description of actual process and measurement noise statistics. This approach is an extension to dissipative systems of the approach described in Ref. [36] for positive real systems.

Linear regulator theory provides the optimal state feedback for minimizing a quadratic objective function as follows. For an LTI system, $\dot{x} = Ax + Bf$, with full

state feedback control law, $f = -C_c x$, the linear regulator problem is to determine the feedback gain, C_c , such that a quadratic objective function

$$J = \int_0^\infty x^T Q_r x + f^T R_r f dt$$

is minimized, where $Q_r = Q_r^T \geq 0$, and $R_r = R_r^T > 0$, are the weighting matrices for state deviations and control effort. The optimal gain is $C_c = R_r^{-1} B^T P_c$, where P_c is the stabilizing solution of the Riccati equation,

$$A^T P_c + P_c A - P_c B R_r^{-1} B^T P_c + Q_r = 0$$

For output feedback controllers, since the system state is not measured, state estimators are required to provide an estimate of the state. If the covariance matrix of Gaussian process noise in the system is $V_f = V_f^T \geq 0$, and the covariance matrix of Gaussian measurement noise is $W_f = W_f^T > 0$, then the optimal Kalman filter for state estimates is given by

$$\dot{\hat{x}} = (A - P_f C^T W_f^{-1} C) \hat{x} + B f + P_f C^T W_f^{-1} y$$

where P_f is the stabilizing solution of the Riccati equation,

$$A P_f + P_f A^T - P_f C^T W_f^{-1} C P_f + V_f = 0$$

so that \hat{x} is an optimal estimate of the state. Combining the estimator with the state feedback linear regulator results in a controller with a realization $(A_c, B_c, C_c, 0)$, where

$$\begin{aligned} A_c &= A - B R_r^{-1} B^T P_c - P_f C^T W_f^{-1} C \\ B_c &= P_f C^T W_f^{-1} \\ C_c &= R_r^{-1} B^T P_c \end{aligned}$$

Using the separation theorem, LQG theory establishes that this controller minimizes the following objective function,

$$J = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{E} \left\{ \int_0^T [x^T Q_r x + f^T R_r f] dt \right\}$$

where $\mathcal{E} \{ \cdot \}$ denotes the estimated value of the argument. Optimality of this objective function holds when the matrices $V_f = V_f^T \geq 0$ and $W_f = W_f^T > 0$, represent noise covariance matrices for process noise and measurement noise, respectively. However,

in the current design approach, these matrices are treated as design parameters for synthesis of state estimators such that the overall controller satisfies desirable dissipativity criteria for robust stability. Therefore, the approach is to select weighting matrices $Q_r = Q_r^T \geq 0$ and $R_r = R_r^T > 0$ for the quadratic objective function, and then design a state estimator, using $V_f = V_f^T \geq 0$ and $W_f = W_f^T > 0$, as design parameters, such that the resulting controller satisfies the dissipativity constraints.

Three controller designs are presented for the spring-mass-damper system to illustrate the application of the stability results of dissipative systems to robust control synthesis. For all these controllers, the weighting matrices for the linear regulator objective function are chosen as $Q_r = 100 * C^T C$ and $R = I$. Design parameters for the state estimator were chosen as $V_f = p_1 C^T C$ and $W_f = p_2 I$. Scalar parameters p_1, p_2 were designed for the controller to satisfy desired robust stability conditions. The performance measure, \mathcal{P} , used for a comparison of these controllers is the \mathcal{H}_2 norm of the closed-loop system, or equivalently, the root mean square (RMS) value of the output, with zero mean, unit intensity white noise applied at the input.

First, a controller is designed while ensuring that its \mathcal{H}_∞ norm satisfies the small gain condition for stability in feedback around the spring-mass-damper system. With $p_1 = 1$ and $p_2 = 15$, a controller is obtained which satisfies the small gain stability condition, that is, its \mathcal{H}_∞ norm is less than $1/2.4788$. This may be verified by solving the dissipativity with a positive definite matrix, P . The performance value, \mathcal{P} , for this controller is 0.809.

Next, a controller is designed such that it is output bounded with respect to the output matrix gain, $\gamma_o = \gamma_i^{-1}$, which guarantees stability for all plants which are input matrix gain bounded with respect to $\gamma_i = \begin{bmatrix} 1.771 & 0.697 \\ 0.697 & 1.820 \end{bmatrix}$, which includes the spring-mass-damper system being considered. It can be verified that such a controller is obtained with $p_1 = 1.0$, and $p_2 = 13$. The performance of this controller is $\mathcal{P} = 0.800$.

Finally, a controller is designed which is inside sector $[-0.3757, 1.718]$. This would guarantee stability of the system with the spring-mass-damper in the feedforward loop, using the sector stability condition, since the plant lies inside sector $[-0.528, 2.662]$. A sector-bounded system, as desired, is obtained with $p_1 = 1.2, p_2 = 15$. The performance of this controller is $\mathcal{P} = 0.798$.

Note that there is not much difference in performance of these controllers. This is because the aim in designing these controllers was to present a strategy for synthesizing robust, dissipative controllers, rather than minimizing the two-norm of the closed-loop system. Synthesis techniques which for robust dissipative controllers that optimize closed-loop performance will be addressed in the future.

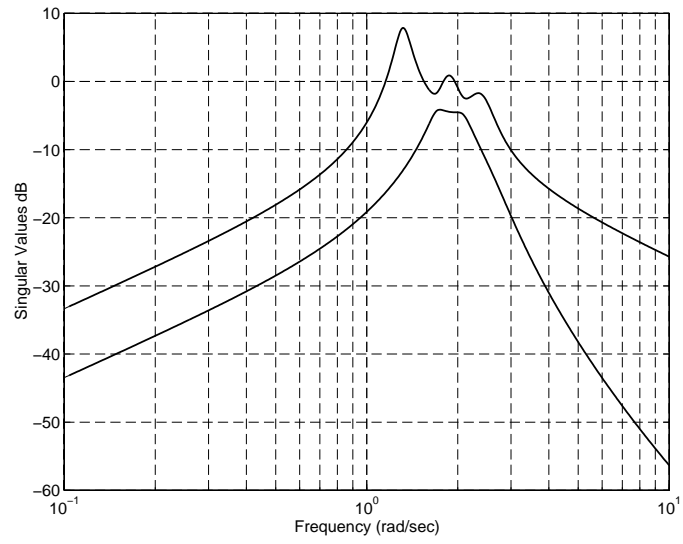


Figure 11.2: Singular Value Plot of the Spring-Mass-Damper System.

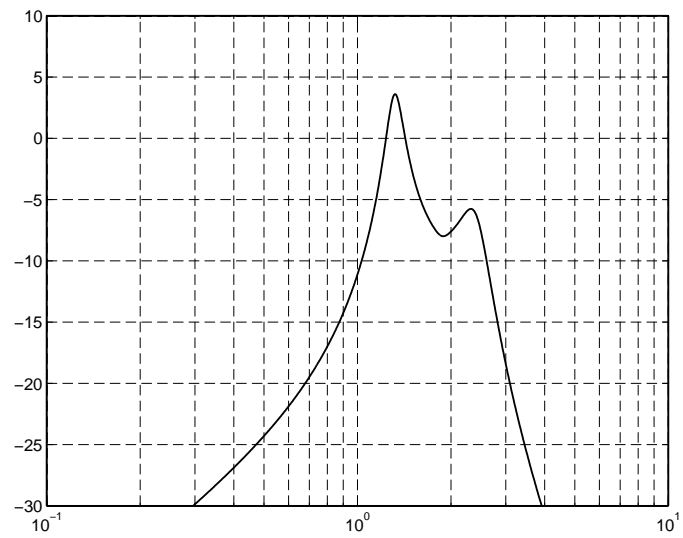


Figure 11.3: Bode Plot of the SISO Model.

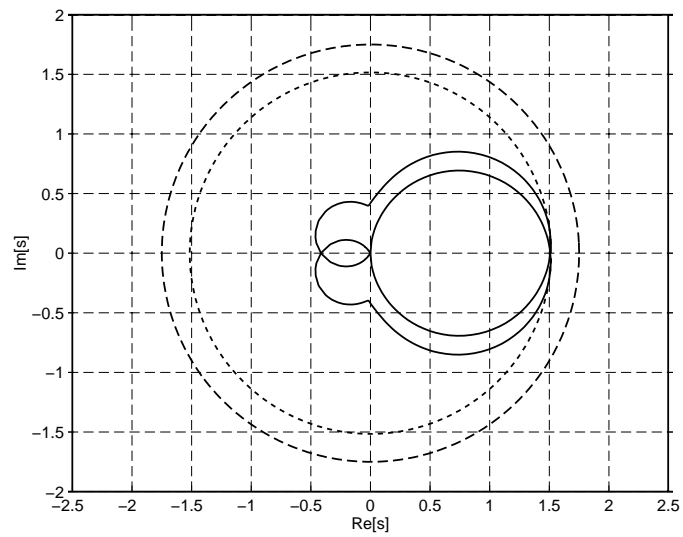


Figure 11.4: Nyquist Plot of the SISO Model.

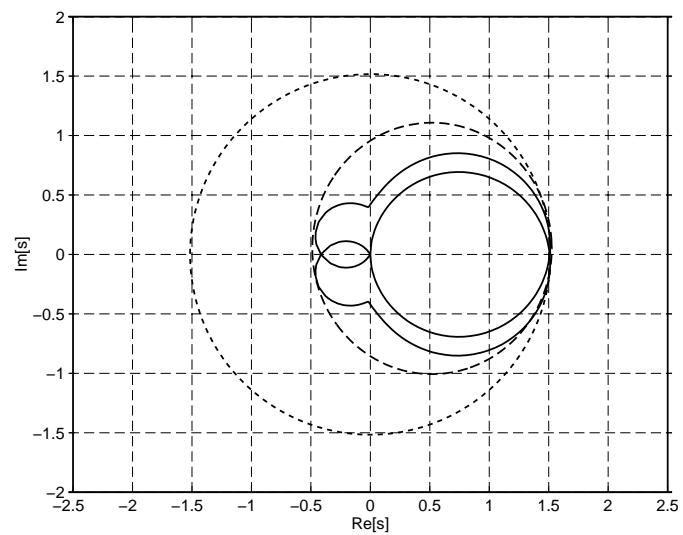


Figure 11.5: Nyquist Plot of the SISO Model, with Smallest Circle.

Chapter 12

Summary

A detailed investigation of linear time-invariant (LTI) systems which are dissipative with respect to quadratic power functions has been presented in this report. In this framework, robust stability results have been developed for a large class of systems, employing mathematical abstractions of the notions of physical power and energy. Gain bounded systems, positive real systems, and sector bounded LTI systems are shown to be dissipative with respect to certain quadratic power functions. Novel concepts of gain-matrices bounded LTI systems have been introduced, and are shown to be a class of dissipative LTI systems. It is demonstrated that dissipative LTI systems represent a large class of LTI systems. Stability results presented for dissipative LTI systems have been developed, unifying and extending a number of stability results available in the literature. Specifically, small gain, positivity, and sector conditions for stability are shown to be special cases of the stability results for dissipative LTI systems; and new stability results for input/output gain-matrices bounded LTI systems have been presented.

State space characterization of dissipative LTI systems has been presented in terms of the dissipativity lemma, which provided a generalization of the bounded realness lemma and the Kalman-Yakubovitch lemma or the positive realness lemma. The state space characterization is equivalently expressed as a linear matrix inequality (LMI) in terms of a minimal state space realization of the LTI system. For certain cases, the LMI characterization has been shown to be equivalent to a quadratic matrix inequality (QMI), which led to an algebraic Riccati equation (ARE) characterization of dissi-

pative LTI systems. Frequency domain characterization of dissipative LTI systems was explored. Necessary conditions for dissipative LTI systems have been presented in terms of frequency domain inequalities (FDIs), and these conditions were shown to be sufficient as well for a large class of dissipative LTI systems. Strictly dissipative LTI systems, which are essential in the development of robust stability results for dissipative systems, are defined as a further restricted class of dissipative LTI systems. Time-domain and frequency-domain characterizations of strictly dissipative LTI systems have also been developed in this report. State space characterizations, and time-domain as well as frequency-domain properties of bounded real, positive real and sector bounded systems have been shown to follow directly from the results of dissipative LTI systems.

The framework of dissipative LTI systems has been employed to develop general robust stability results. In particular, three stability results involving dissipative LTI systems have been presented in this report. Sufficient conditions were presented for (1) stability of the feedback interconnection of dissipative LTI systems, (2) stability of dissipative LTI systems with memoryless feedback nonlinearities, and (3) quadratic stability of uncertain linear systems. The Lyapunov function approach has been used to establish these results, with the energy functions of the dissipative LTI systems being the Lyapunov functions. Stability conditions for these problems, derived from small gain, positivity and sector criteria, were shown to be special cases of the results for dissipative LTI systems. New stability results for feedback interconnection of LTI systems, in terms of input/output gain-matrix bounded LTI systems were also shown to follow as special cases of the stability results for dissipative LTI systems. Thus, stability results for dissipative LTI system have been shown to be general results, which unify and extend a number of stability results from the literature.

Numerical techniques for tight characterization of given LTI systems, in terms of dissipativity with respect to quadratic power functions, have also been presented in the report. This approach utilized recently developed positive semidefinite programming techniques to solve linear matrix inequalities. A number of formulations have been presented for selection of power functions with prescribed structure. Robust controller synthesis techniques, based on the stability results for dissipative LTI systems, have been discussed. In particular, an approach for dissipative controller synthesis, employing optimal linear regulators and state estimators, has been presented. The state estimators were designed such that the overall compensator is dissipative with respect to required power functions for robust stability. A numerical example of a

spring-mass-damper system has been employed for a demonstration of the application of results presented in this report.

Future work would involve further investigation into approaches for tight characterization of uncertain plants, with parametric uncertainty and structured uncertainties. Also, robust controller synthesis techniques which enhance overall system performance need to be developed further. Finally, characterization of nonlinear systems which are dissipative with respect to specified power functions, and development of specific stability results for these systems could be pursued in the future.

Appendix A

Signals and Systems

This appendix summarizes some results from signals and systems theory used in this report. First, function spaces for the input and output signals are described, and then the representations of linear time-invariant (LTI) systems in the state space form and the frequency-domain, along with certain operations on these systems, are presented. Properties of bounded real, positive real, and sector bounded LTI systems are reviewed, and bilinear transformations between these systems are presented. The last part of this appendix discusses the spectral factorization theorem, which is used in the frequency-domain characterization of dissipative LTI systems.

Extended spaces of square-integrable functions form the mathematical framework for the input and the output signals of continuous-time systems. The space of (Lebesgue) square-integrable functions, that is, real-valued functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ which satisfy $\int_0^\infty f^T(t)f(t) dt < \infty$, will be denoted as \mathcal{L}_2^m . An inner product on this space is defined as $\langle y, f \rangle = \int_0^\infty y^T(t)f(t) dt$, for all $y, f \in \mathcal{L}_2^m$. With this inner product, \mathcal{L}_2^m is a Hilbert space, and the natural norm, induced by the inner product, is expressed as $\| f \| = [\langle f, f \rangle]^{1/2}$. The Fourier transforms of signals in \mathcal{L}_2^m also form a Hilbert space. This space, denoted by $\mathcal{L}_{2+}^m(j\mathbb{R})$, is a space of complex functions, $\tilde{f} : \mathcal{C} \rightarrow \mathcal{C}^m$, which are analytic in the closed, right-half plane and satisfy $\int_{-\infty}^\infty \tilde{f}^*(j\omega)\tilde{f}(j\omega) d\omega < \infty$. The inner product in this space is $\langle \tilde{y}, \tilde{f} \rangle_{\mathcal{L}_{2+}^m(j\mathbb{R})} = \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{y}^*(j\omega)\tilde{f}(j\omega) d\omega$, and the induced norm in this space is $\| f \| = [\langle \tilde{f}, \tilde{f} \rangle]^{1/2}$. The subspace of real, rational, proper functions in $\mathcal{L}_{2+}^m(j\mathbb{R})$ is denoted as $\mathcal{RL}_{2+}^m(j\mathbb{R})$. Matrices with real, rational, proper elements which are analytic in

the closed, right-half plane, form an inner product space denoted by $\mathcal{RL}_2^{p \times m}(j\mathbb{R})$. The extended Parseval's theorem states that for any $f, y \in \mathcal{L}_2^m$, the inner product $\langle y, f \rangle = \int_0^\infty y^T(t)f(t)dt = \int_{-\infty}^\infty \tilde{y}^*(j\omega)\tilde{f}(j\omega)d\omega = \langle \tilde{y}, \tilde{f} \rangle$, where $\tilde{y}, \tilde{f} \in \mathcal{L}_{2+}^m(j\mathbb{R})$ are Fourier transforms of y, f respectively, and vice versa. Thus, the Fourier transform provides a an isometric isomorphism between \mathcal{L}_2^m and $\mathcal{L}_{2+}^m(j\mathbb{R})$.

In response to “well-behaved” input functions, the output of unstable dynamic systems may increase without bound as time increases; specifically, for the inputs in \mathcal{L}_2^m , the outputs may not be \mathcal{L}_2^m . In fact, one definition of the stability of a dynamic system (bounded input, bounded output stability) is that the outputs be in \mathcal{L}_2^m for all inputs from \mathcal{L}_2^m . Therefore, for the study of the stability of dynamic systems, a notion of extended spaces, which contain both the “well-behaved” signals and the “exploding” signals, is needed.

The truncation operator or truncation projection is needed to define extended spaces. Given any signal, $f : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, the truncated signal, denoted as $f_T(t)$, for $T \in [0, \infty)$, is defined as

$$f_T(t) = \begin{cases} f(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases}$$

This is a mathematical statement of the intuitive concept of truncating a signal at time T . The extended space, corresponding to the space \mathcal{L}_2^m , denoted by \mathcal{L}_{2e}^m , is defined as follows

$$\mathcal{L}_{2e}^m = \{f \mid f : \mathbb{R}_+ \rightarrow \mathbb{R}^m, f_T \in \mathcal{L}_2^m \quad \forall \quad T \in [0, \infty)\}.$$

Note that \mathcal{L}_{2e}^m is only a linear space; that is, it is not an inner product space or a normed space. For all $f \in \mathcal{L}_2^m$, it follows that $f_T \in \mathcal{L}_2^m$, for all $T \in [0, \infty)$. Thus, \mathcal{L}_2^m is a subspace within \mathcal{L}_{2e}^m . Also, given any function, $f : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, if $f_T \in \mathcal{L}_2^m$ for all $T \in [0, \infty)$, then $\|f_T\|$ is a nondecreasing function of T . In this case, if $\lim_{T \rightarrow \infty} \|f_T\|$ exists, then $f \in \mathcal{L}_2^m$, and $\lim_{T \rightarrow \infty} \|f_T\| = \|f\|$. Thus, this extended space contains both “well-behaved” functions as well as “exploding” functions. For example, $f_1 = e^{\sigma t}$ for $\sigma < 0$ is in both spaces, \mathcal{L}_2^m and \mathcal{L}_{2e}^m ; however, $f_2 = e^{\sigma t}$ for $\sigma \geq 0$ belongs to \mathcal{L}_{2e}^m , but does not belong to \mathcal{L}_2^m . The extended space of square integrable functions, \mathcal{L}_{2e}^m , is the universal set for inputs and outputs of continuous time LTI systems examined in this work.

Next some properties of linear, time-invariant (LTI) systems are reviewed. A state

space realization of a linear, time-invariant system, Σ , is given by

$$\begin{aligned}\dot{x} &= Ax + Bf \\ y &= Cx + Df\end{aligned}$$

where $y(t)$ is the $p \times 1$ output vector, $f(t)$ is an $m \times 1$ input vector, $x(t)$ is an $n \times 1$ state vector and the system matrices (A, B, C, D) describe the dynamics of the LTI system. The $p \times m$ transfer function matrix for this system is $G(s) = C(sI - A)^{-1}B + D$. The impulse response matrix for the system, Σ , is given by $G(t) = Ce^{At}B + D\delta(t)$. This system is stable if and only if the eigenvalues of A are in the open left-half plane. If A is a stable matrix, then $G(s) \in \mathcal{RL}_{2+}^{p \times m}(j\mathfrak{R})$. A state space realization, (A, B, C, D) , is a minimal realization if and only if (A, B) is controllable and (A, C) is observable. If $z = Tx$, where T is a nonsingular, state transformation matrix, the transformed state space realization of $G(s)$ is given as $(T^{-1}AT, T^{-1}B, CT, D)$. Two minimal state space realizations of a transfer function matrix are related by a state space transformation. Paraconjugate transpose of a transfer function matrix, $G(s)$, is given by $G^\sim(s) = G^T(-s) = B^T(-sI - A^T)^{-1}C^T + D^T$. Thus, if (A, B, C, D) is a state space realization of $G(s)$, then $(-A^T, -C^T, B^T, D^T)$ and $(-A^T, C^T, -B^T, D^T)$ are state space realizations of $G^\sim(s)$.

Next, consider state space realizations for parallel, series and feedback interconnections of two LTI systems, in terms of their individual realizations. Let (A_1, B_1, C_1, D_1) be a state space realization of $G_1(s)$ and (A_2, B_2, C_2, D_2) be a state space realization of $G_2(s)$. A state space realization of the parallel connection of $G_1(s)$ and $G_2(s)$, that is, a state space realization of $G_1(s) + G_2(s)$ is

$$\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D_1 + D_2 \right)$$

A state space realization of the series connection of $G_1(s)$ and $G_2(s)$, that is, $G_1(s)G_2(s)$, is

$$\left(\begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix}, D_1D_2 \right)$$

Another realization for the series interconnection is

$$\left(\begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix}, \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix}, \begin{bmatrix} D_1C_2 & C_1 \end{bmatrix}, D_1D_2 \right)$$

Finally, the closed-loop transfer function of the negative feedback interconnection of LTI systems $G_1(s)$ and $G_2(s)$ is $T(s) = G_1(s)[I + G_2(s)G_1(s)]^{-1}$. A state space

representation for this interconnection in terms of the states of $G_1(s)$ and $G_2(s)$ is given by $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$, where

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A_1 - B_1 D_2 (I + D_1 D_2)^{-1} C_1 & -B_1 (I + D_2 D_1)^{-1} C_2 \\ B_2 (I + D_1 D_2)^{-1} & A_2 - B_2 D_1 (I + D_2 D_1)^{-1} C_2 \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} B_1 (I + D_2 D_1)^{-1} \\ B_2 D_1 (I + D_2 D_1)^{-1} \end{bmatrix} \\ C_{cl} &= \begin{bmatrix} (I + D_1 D_2)^{-1} C_1 & -(I + D_1 D_2)^{-1} D_1 C_2 \end{bmatrix} \\ D_{cl} &= D_1 (I + D_2 D_1)^{-1} \end{aligned}$$

Properties of bounded real, positive real and sector bounded systems are discussed next. Bounded real systems are systems with finite \mathcal{H}_∞ norm [9, 10]. Consider the systems with unity gain, that is, $\|G(s)\|_\infty \leq 1$. Recall that \mathcal{H}_∞ norm of a system is the induced operator norm with the \mathcal{L}_2^m norm for the input, f , and the output, y , [30]. Therefore, the condition for \mathcal{H}_∞ norm of a system being bounded by unity implies that

$$\int_0^T y^T(t)y(t)dt \leq \int_0^\infty y^T(t)y(t)dt \leq \int_0^\infty f_T^T(t)f_T(t)dt = \int_0^T f^T(t)f(t)dt$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$, with $y(t)$ being the system response to the truncated input, $f_T(t)$. Thus bounded real systems satisfy

$$\int_0^T \{f^T(t)f(t) - y^T(t)y(t)\} dt \geq 0,$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. In the frequency-domain, an LTI system with transfer function, $G(s)$, is bounded real if $I - G^*(j\omega)G(j\omega) \geq 0$ for all ω . For single-input, single-output systems, the bounded realness condition can be visualized in the frequency-domain as the frequency response being within a unit circle centered at the origin in the frequency plane. A system is strictly bounded real if it satisfies the conditions for bounded realness in a strict sense. Small gain conditions for stability state that the feedback interconnection of a bounded real systems and a strictly bounded real system is stable.

Passive systems are characterized by the input-output property $\int_0^T y^T(t)f(t)dt \geq 0$, for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$ [11]. Equivalently, passive systems satisfy

$$\int_0^T \{y^T(t)f(t) + f^T(t)y(t)\} dt \geq 0,$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. In the frequency-domain, an LTI system is passive, or equivalently, the transfer function is positive real, if $G^*(j\omega) + G(j\omega) \geq 0$, for all ω . Strictly positive real systems satisfy these conditions in a strict sense. Passivity conditions for stability state that the feedback interconnection of a positive real system and a strictly positive real systems is stable.

A number of sector boundedness conditions for LTI systems are described in the literature. An LTI system inside sector $[a, b]$, with $\infty > b > a$, satisfies $\langle (y - af), (y - bf) \rangle_T \leq 0$, for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$ [13, 29]. A memoryless system is inside sector $[a, b]$ if its graph lies within a conical region in the input-output space defined by this inequality. If the memoryless system is time-varying, then the shape of the graph of a time-varying nonlinearity changes shape with time, however, it must stay within this conical region for all time, if the nonlinearity is sector bounded. For an LTI system, the sector boundedness condition may be rewritten as $\int_0^T (y(t) - af(t))^T (y(t) - bf(t)) dt \leq 0$, or, equivalently,

$$\int_0^T \left\{ -ab f^T(t) f(t) + (a + b) y^T(t) f(t) - y^T(t) y(t) \right\} dt \geq 0,$$

for all $T \in [0, \infty)$ and $f \in \mathcal{L}_{2e}^m$. In the frequency-domain, a transfer function, $G(s)$, is inside sector $[a, b]$ if $\text{herm}\{[G(j\omega) - aI]^* [G(j\omega) - bI]\} \leq 0$, for all ω , where $\text{herm}(\cdot)$ stands for Hermitian part of the argument, that is, $\text{herm}(M) = 0.5(M^* + M)$. The frequency plane provides a simple visualization for sector bounded SISO systems. The frequency response of a SISO system inside sector $[a, b]$ lies within a circle in the frequency plane, which is centered on the real axis and intersects the real axis at a and b . LTI systems that satisfy these conditions in a strict sense are strictly inside sector $[a, b]$. For $b > 0 > a$, a sector stability result states that the feedback interconnection of an LTI system inside sector $[a, b]$ with another LTI system which is strictly inside sector $[-\frac{1}{b}, -\frac{1}{a}]$ is stable.

Bilinear transformations between positive real systems, bounded real systems and sector bounded systems are reviewed next [9, 10, 12]. Let S be a bounded real system, so that $y = Sf$ satisfies $\langle y, y \rangle - \langle f, f \rangle \leq 0$. Noting that $\langle f, f \rangle - \langle y, y \rangle = \langle (f - y), (f + y) \rangle \geq 0$, it follows that $Z = (I - S)(I + S)^{-1}$ is positive real. Conversely, if Z is positive real, then $S = (I - Z)(I + Z)^{-1}$ is bounded real. Further, if a system, T , is inside sector $[a, b]$, then the system, $S = r^{-1}(T - \alpha I)$, where $r = (b - a)/2$ and $\alpha = (a + b)/2$, is bounded real [37]. Conversely, if S is bounded real, then $T = rS + \alpha I$ is inside sector $[a, b]$. This fact can be derived from the following manipulations. Let

$y = Tf$ and $y' = Sf$; then, $y' = r^{-1}(y - \alpha f)$, and

$$\begin{aligned}
\langle y', y' \rangle - \langle f, f \rangle &= \frac{1}{r^2} \{ \langle y - \alpha f, y - \alpha f \rangle - r^2 \langle f, f \rangle \} \\
&= \frac{1}{r^2} \{ \langle y, y \rangle - 2\alpha \langle y, f \rangle + (\alpha^2 - r^2) \langle f, f \rangle \} \\
&= \frac{1}{r^2} \{ \langle y, y \rangle - (a + b) \langle y, f \rangle + ab \langle f, f \rangle \} \\
&= \frac{1}{r^2} \{ \langle y - af, y - bf \rangle \}
\end{aligned}$$

This shows that $\langle y', y' \rangle - \langle f, f \rangle \leq 0$ if and only if $\langle y - af, y - bf \rangle \leq 0$, hence the result.

Finally, the spectral factorization theorem is discussed, since it is used in the development of frequency-domain characterization of dissipative LTI systems. A transfer function matrix, $\Phi(s)$, is called a parahermitian matrix if it satisfies $\Phi^\sim(s) = \Phi(s)$. The spectral factorization theorem essentially states that a parahermitian transfer function matrix, which is positive semidefinite on the imaginary axis, can be factorized with stable factors. This may be thought of as an extension of the concept of Cholesky decomposition of positive semidefinite matrices. The theorem states that a given parahermitian transfer function matrix $\Phi^\sim(s) = \Phi(s)$, which satisfies $\Phi(j\omega) \geq 0$, for all $\omega \in \Re$, can be factorized as $\Phi(s) = M^\sim(s)M(s)$, where $M(s)$ is a stable transfer function matrix with transmission zeros in the closed left-half plane.

This appendix has presented the notation and some results which have been used in this work.

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